Trigonometric Graphs and Identities

BIG Ideas
- Graph trigonometric functions and determine period, amplitude, phase shifts, and vertical shifts.
- Use and verify trigonometric identities.
- Solve trigonometric equations.

Key Vocabulary
- amplitude (p. 823)
- phase shift (p. 829)
- vertical shift (p. 831)
- trigonometric identity (p. 837)
- trigonometric equation (p. 861)

Real-World Link
Music String vibrations produce the sound you hear in stringed instruments such as guitars, violins, and pianos. These vibrations can be modeled using trigonometric functions.

Foldables Study Organizer
Trigonometric Graphs and Identities Make this Foldable to help you organize your notes. Begin with eight sheets of grid paper.

1. Staple the stack of grid paper along the top to form a booklet.
2. Cut seven lines from the bottom of the top sheet, six lines from the second sheet, and so on. Label with lesson numbers as shown.
GET READY for Chapter 14
Diagnose Readiness You have two options for checking Prerequisite Skills.

Option 2
Take the Online Readiness Quiz at algebra2.com.

Option 1
Take the Quick Check below. Refer to the Quick Review for help.

Find the exact value of each trigonometric function. (Lesson 13-3)
1. \( \sin 135^\circ \)
2. \( \tan 315^\circ \)
3. \( \cos 90^\circ \)
4. \( \tan 45^\circ \)
5. \( \sin \frac{\pi}{4} \)
6. \( \cos \frac{7\pi}{6} \)
7. \( \cos (-150^\circ) \)
8. \( \cot \frac{\pi}{4} \)
9. \( \sec \frac{13\pi}{6} \)
10. \( \tan (-\frac{3\pi}{2}) \)
11. \( \tan \frac{8\pi}{3} \)
12. \( \csc (-720^\circ) \)

Example 1
Find the exact value of \( \sin \frac{11\pi}{6} \).
The terminal side of \( \frac{11\pi}{6} \) lies in Quadrant IV, so the reference angle \( \theta \) is \( 2\pi - \frac{11\pi}{6} \) or \( \frac{\pi}{6} \). The sine function is negative in the Quadrant IV.
\[
\sin \frac{11\pi}{6} = -\sin \frac{\pi}{6} = -\frac{1}{2}
\]

Example 2
Factor \( x^3 - 4x^2 - 21x \) completely.
\[
x^3 - 4x^2 - 21x = x(x^2 - 4x - 21)
\]
The product of the coefficients of the \( x \)-terms must be \(-21\), and their sum must be \(-4\). The product of 7 and 3 is 21 and their difference is 4. Since the sum must be negative, the coefficients of the \( x \)-terms are \(-7\) and 3.
\[
x(x^2 - 4x - 21) = x(x - 7)(x + 3)
\]

Example 3
Solve the equation factored in Example 2.
From Example 2,
\[
x^3 - 4x^2 - 21x = x(x - 7)(x + 3).
\]
Apply the Zero Product Property and solve.
\[
x = 0 \quad \text{or} \quad x - 7 = 0 \quad \text{or} \quad x + 3 = 0
\]
\[
x = 7 \quad \text{or} \quad x = -3
\]
The solution set is \(-3, 0, 7\).

Factor completely. If the polynomial is not factorable, write prime. (Lesson 6-6)
14. \(-15x^2 - 5x\)
15. \(2x^4 - 4x^2\)
16. \(x^3 + 4\)
17. \(2x^2 - 3x - 2\)

Example 13
AMUSEMENT The distance from the highest point of a Ferris wheel to the ground can be found by multiplying 60 ft by \( \sin 90^\circ \). What is the height of the Ferris wheel at the highest point? (Lesson 13-3)

Factor completely. If the polynomial is not factorable, write prime. (Lesson 6-6)
14. \(-15x^2 - 5x\)
15. \(2x^4 - 4x^2\)
16. \(x^3 + 4\)
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Graphing Trigonometric Functions

The rise and fall of tides can have great impact on the communities and ecosystems that depend upon them. One type of tide is a semidiurnal tide. This means that bodies of water, like the Atlantic Ocean, have two high tides and two low tides a day. Because tides are periodic, they behave the same way each day.

Graph Trigonometric Functions

The diagram below illustrates the water level as a function of time for a body of water with semidiurnal tides. In each cycle of high and low tides, the pattern repeats itself. Recall that a function whose graph repeats a basic pattern is said to be periodic.

To find the period, start from any point on the graph and proceed to the right until the pattern begins to repeat. The simplest approach is to begin at the origin. Notice that after about 12 hours the graph begins to repeat. Thus, the period of the function is about 12 hours.

To graph the functions \( y = \sin \theta \), \( y = \cos \theta \), or \( y = \tan \theta \), use values of \( \theta \) expressed either in degrees or radians. Ordered pairs for points on these graphs are of the form \((\theta, \sin \theta)\), \((\theta, \cos \theta)\), and \((\theta, \tan \theta)\), respectively.

**Review Vocabulary**

**Period** The least possible value of \( a \) for which \( f(x) = f(x + a) \).

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<thead>
<tr>
<th>( \theta )</th>
<th>( 0^\circ )</th>
<th>( 30^\circ )</th>
<th>( 45^\circ )</th>
<th>( 60^\circ )</th>
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After plotting several points, complete the graphs of \( y = \sin \theta \) and \( y = \cos \theta \) by connecting the points with a smooth, continuous curve. Recall from Chapter 13 that each of these functions has a period of 360° or \( 2\pi \) radians. That is, the graph of each function repeats itself every 360° or \( 2\pi \) radians.

Notice that both the sine and cosine have a maximum value of 1 and a minimum value of -1. The amplitude of the graph of a periodic function is the absolute value of half the difference between its maximum value and its minimum value. So, for both the sine and cosine functions, the amplitude of their graphs is \( \left| \frac{1 - (-1)}{2} \right| = 1 \).

By examining the values for \( \tan \theta \) in the table, you can see that the tangent function is not defined for 90°, 270°, ..., 90° + \( k \cdot 180° \), where \( k \) is an integer. The graph is separated by vertical asymptotes whose x-intercepts are the values for which \( y = \tan \theta \) is not defined.

The period of the tangent function is 180° or \( \pi \) radians. Since the tangent function has no maximum or minimum value, it has no amplitude.

Compare the graphs of the secant, cosecant, and cotangent functions to the graphs of the cosine, sine, and tangent functions, shown below.

Notice that the period of the secant and cosecant functions is 360° or \( 2\pi \) radians. The period of the cotangent is 180° or \( \pi \) radians. Since none of these functions have a maximum or minimum value, they have no amplitude.
Variations of Trigonometric Functions  Just as with other functions, a trigonometric function can be used to form a family of graphs by changing the period and amplitude.

**Period and Amplitude**

On a TI-83/84 Plus, set the MODE to degrees.

**THINK AND DISCUSS**

1. Graph \( y = \sin x \) and \( y = \sin 2x \). What is the maximum value of each function?

2. How many times does each function reach a maximum value?

3. Graph \( y = \sin \left( \frac{x}{2} \right) \). What is the maximum value of this function? How many times does this function reach its maximum value?

4. Use the equations \( y = \sin bx \) and \( y = \cos bx \). Repeat Exercises 1–3 for maximum values and the other values of \( b \). What conjecture can you make about the effect of \( b \) on the maximum values and the periods of these functions?

5. Graph \( y = \sin x \) and \( y = 2 \sin x \). What is the maximum value of this function? How many times does this function reach its maximum value?

6. Graph \( y = \frac{1}{2} \sin x \). What is the maximum value of this function? What is the period of this function?

7. Use the equations \( y = a \sin x \) and \( y = a \cos x \). Repeat Exercises 5 and 6 for other values of \( a \). What conjecture can you make about the effect of \( a \) on the amplitudes and periods of \( y = a \sin x \) and \( y = a \cos x \)?

The results of the investigation suggest the following generalization.
You can use the amplitude and period of a trigonometric function to help you graph the function.

**EXAMPLE**  
**Graph Trigonometric Functions**

Find the amplitude, if it exists, and period of each function. Then graph the function.

**a.** \( y = \cos 3\theta \)

First, find the amplitude.

\[
|a| = |1| \quad \text{The coefficient of } \cos 3\theta \text{ is } 1.
\]

Next, find the period.

\[
\frac{360^\circ}{|b|} = \frac{360^\circ}{13} \quad b = 3
\]

\[
= 120^\circ
\]

Use the amplitude and period to graph the function.

**b.** \( y = \tan \left(-\frac{1}{3}\theta\right) \)

Amplitude: This function does not have an amplitude because it has no maximum or minimum value.

Period:

\[
\frac{\pi}{|b|} = \frac{\pi}{-\frac{1}{3}}
\]

\[
= 3\pi
\]
1. \( y = \frac{1}{2} \sin \theta \)
2. \( y = 2 \sin \theta \)
3. \( y = \frac{2}{3} \cos \theta \)
4. \( y = \frac{1}{4} \tan \theta \)
5. \( y = \csc 2\theta \)
6. \( y = 4 \sin 2\theta \)
7. \( y = 4 \cos \frac{3}{4} \theta \)
8. \( y = \frac{1}{2} \sec 3\theta \)
9. \( y = \frac{3}{4} \cos \frac{1}{2} \theta \)

**BIOLOGY** For Exercises 10 and 11, use the following information.
In a certain wildlife refuge, the population of field mice can be modeled by \( y = 3000 + 1250 \sin \frac{\pi}{6} t \), where \( y \) represents the number of mice and \( t \) represents the number of months past March 1 of a given year.

10. Determine the period of the function. What does this period represent?

11. What is the maximum number of mice, and when does this occur?
Find the amplitude, if it exists, and period of each function. Then graph each function.

12. \(y = 3 \sin \theta\)  
13. \(y = 5 \cos \theta\)  
14. \(y = 2 \csc \theta\)

15. \(y = 2 \tan \theta\)  
16. \(y = \frac{1}{5} \sin \theta\)  
17. \(y = \frac{1}{3} \sec \theta\)

18. \(y = \sin 4\theta\)  
19. \(y = \sin 2\theta\)  
20. \(y = \sec 3\theta\)

21. \(y = \cot 5\theta\)  
22. \(y = 4 \tan \frac{1}{3}\theta\)

23. \(y = 2 \cot \frac{1}{2}\theta\)

**MEDICINE** For Exercises 24 and 25, use the following information.
Doctors may use a tuning fork that resonates at a given frequency as an aid to diagnose hearing problems. The sound wave produced by a tuning fork can be modeled using a sine function.

24. If the amplitude of the sine function is 0.25, write the equations for tuning forks that resonate with a frequency of 64, 256, and 512 Hertz.

25. How do the periods of the tuning forks compare?

Find the amplitude, if it exists, and period of each function. Then graph each function.

26. \(y = 6 \sin \frac{2}{3}\theta\)  
27. \(y = 3 \cos \frac{1}{2}\theta\)  
28. \(y = 3 \csc \frac{1}{2}\theta\)

29. \(y = \frac{1}{2} \cot 2\theta\)  
30. \(2y = \tan \theta\)

31. \(\frac{3}{4}y = \frac{2}{3} \sin \frac{3}{5}\theta\)

32. Draw a graph of a sine function with an amplitude \(\frac{3}{5}\) and a period of 90°. Then write an equation for the function.

33. Draw a graph of a cosine function with an amplitude of \(\frac{7}{8}\) and a period of \(\frac{2\pi}{5}\). Then write an equation for the function.

34. Graph the functions \(f(x) = \sin x\) and \(g(x) = \cos x\), where \(x\) is measured in radians, for \(x\) between 0 and \(2\pi\). Identify the points of intersection of the two graphs.

35. Identify all asymptotes to the graph of \(g(x) = \sec x\).

**BOATING** For Exercises 36–38, use the following information.
A marker buoy off the coast of Gulfport, Mississippi, bobs up and down with the waves. The distance between the highest and lowest point is 4 feet. The buoy moves from its highest point to its lowest point and back to its highest point every 10 seconds.

36. Write an equation for the motion of the buoy. Assume that it is at equilibrium at \(t = 0\) and that it is on the way up from the normal water level.

37. Draw a graph showing the height of the buoy as a function of time.

38. What is the height of the buoy after 12 seconds?

39. **OPEN ENDED** Write a trigonometric function that has an amplitude of 3 and a period of \(\pi\). Graph the function.

40. **REASONING** Explain what it means to say that the period of a function is 180°.

41. **CHALLENGE** A function is called even if the graphs of \(y = f(x)\) and \(y = f(-x)\) are exactly the same. Which of the six trigonometric functions are even? Justify your answer with a graph of each function.
42. **FIND THE ERROR** Dante and Jamile graphed \( y = 3 \cos \frac{2}{3} \theta \). Who is correct? Explain your reasoning.

43. **Writing in Math** Use the information on page 822 to explain how you can predict the behavior of tides. Explain why certain tidal characteristics follow the patterns seen in the graph of the sine function.

44. **ACT/SAT** Identify the equation of the graphed function.

45. **REVIEW** Refer to the figure below. If \( \tan x = \frac{10}{24} \), what are \( \sin x \) and \( \cos x \)?

   - **A** \( \sin x = \frac{1}{2} \) sin 4\( \theta \)  
   - **B** \( \sin x = \frac{1}{4} \) sin 2\( \theta \)  
   - **C** \( \sin x = 2 \) sin \( \frac{1}{4} \) \( \theta \)  
   - **D** \( \sin x = 4 \) sin \( \frac{1}{2} \) \( \theta \)

46. \( x = \sin^{-1} 1 \)  
47. \( \arcsin (-1) = y \)  
48. \( \arccos \sqrt{\frac{2}{2}} = x \)

49. \( \sin 390^\circ \)  
50. \( \sin (-315^\circ) \)  
51. \( \cos 405^\circ \)

52. **PROBABILITY** There are 8 girls and 8 boys on the Faculty Advisory Board. Three are juniors. Find the probability of selecting a boy or a girl from the committee who is not a junior. (Lesson 12-5)

53. Find the first five terms of the sequence in which \( a_1 = 3 \), \( a_{n+1} = 2a_n + 5 \). (Lesson 11-5)

54. \( y = x^2, y = 3x^2 \)  
55. \( y = 3x^2, y = 3x^2 - 4 \)  
56. \( y = 2x^2, y = 2(x + 1)^2 \)
Main Ideas
- Graph horizontal translations of trigonometric graphs and find phase shifts.
- Graph vertical translations of trigonometric graphs.

New Vocabulary
phase shift
c Vertical shift
c Midline

In predator-prey ecosystems, the number of predators and the number of prey tend to vary in a periodic manner. In a certain region with coyotes as predators and rabbits as prey, the rabbit population $R$ can be modeled by the equation

$$R = 1200 + 250 \sin \frac{1}{2} \pi t,$$

where $t$ is the time in years since January 1, 2001.

**Horizontal Translations** Recall that a translation is a type of transformation in which the image is identical to the preimage in all aspects except its location on the coordinate plane. A horizontal translation shifts to the left or right, and not upward or downward.

**GRAPHING CALCULATOR**

**Horizontal Translations**

On a TI-83/84 Plus, set the MODE to degrees.

**THINK AND DISCUSS**

1. Graph $y = \sin x$ and $y = \sin (x - 30)$. How do the two graphs compare?
2. Graph $y = \sin (x + 60)$. How does this graph compare to the other two?
3. What conjecture can you make about the effect of $h$ in the function $y = \sin (x - h)$?
4. Test your conjecture on the following pairs of graphs.
   - $y = \cos x$ and $y = \cos (x + 30)$
   - $y = \tan x$ and $y = \tan (x - 45)$
   - $y = \sec x$ and $y = \sec (x + 75)$

Notice that when a constant is added to an angle measure in a trigonometric function, the graph is shifted to the left or to the right. If $(x, y)$ are coordinates of $y = \sin x$, then $(x \pm h, y)$ are coordinates of $y = \sin (x \pm h)$. A horizontal translation of a trigonometric function is called a **phase shift**.
The secant, cosecant, and cotangent can be graphed using the same rules.

**Words**

The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is $h$, where $b > 0$.

If $h > 0$, the shift is to the right.
If $h < 0$, the shift is to the left.

**Models:**

**Sine**

$y = a \sin b(\theta - h); h > 0$

$y = a \sin b(\theta - h); h < 0$

**Cosine**

$y = a \cos b(\theta - h); h > 0$

$y = a \cos b(\theta - h); h < 0$

**Tangent**

$y = a \tan b(\theta - h); h > 0$

$y = a \tan b(\theta - h); h < 0$

The phase shift of the functions $y = a \sin b(\theta - h)$, $y = a \cos b(\theta - h)$, and $y = a \tan b(\theta - h)$ is $h$, where $b > 0$.

If $h > 0$, the shift is to the right.
If $h < 0$, the shift is to the left.

**Example**

**Graph Horizontal Translations**

State the amplitude, period, and phase shift for $y = \cos (\theta - 60^\circ)$. Then graph the function.

Since $a = 1$ and $b = 1$, the amplitude and period of the function are the same as $y = \cos \theta$. However, $h = 60^\circ$, so the phase shift is $60^\circ$. Because $h > 0$, the parent graph is shifted to the right.

To graph $y = \cos (\theta - 60^\circ)$, consider the graph of $y = \cos \theta$. Graph this function and then shift the graph $60^\circ$ to the right. The graph $y = \cos (\theta - 60^\circ)$ is the graph of $y = \cos \theta$ shifted to the right.

1. State the amplitude, period, and phase shift for $y = 2 \sin \left( \theta + \frac{\pi}{4} \right)$. Then graph the function.
**Vertical Translations** In Chapter 5, you learned that the graph of \( y = x^2 + 4 \) is a vertical translation of the parent graph of \( y = x^2 \). Similarly, graphs of trigonometric functions can be translated vertically through a **vertical shift**.

When a constant is added to a trigonometric function, the graph is shifted upward or downward. If \((x, y)\) are coordinates of \( y = \sin x \), then \((x, y \pm k)\) are coordinates of \( y = \sin x \pm k \).

A new horizontal axis called the **midline** becomes the reference line about which the graph oscillates. For the graph of \( y = \sin \theta + k \), the midline is the graph of \( y = k \).

**KEY CONCEPT**

**Vertical Shift**

**Words** The vertical shift of the functions \( y = a \sin b(\theta - h) + k \), \( y = a \cos b(\theta - h) + k \), and \( y = a \tan b(\theta - h) + k \) is \( k \).

If \( k > 0 \), the shift is up. If \( k < 0 \), the shift is down. The midline is \( y = k \).

**Models:**

**Sine**

\[
y = a \sin b(\theta - h) + k; k > 0
\]

\[
y = \sin \theta
\]

\[
y = a \sin b(\theta - h) + k; k < 0
\]

**Cosine**

\[
y = a \cos b(\theta - h) + k; k > 0
\]

\[
y = \cos \theta
\]

\[
y = a \cos b(\theta - h) + k; k < 0
\]

**Tangent**

\[
y = a \tan b(\theta - h) + k; k > 0
\]

\[
y = \tan \theta
\]

\[
y = a \tan b(\theta - h) + k; k < 0
\]

The secant, cosecant, and cotangent can be graphed using the same rules.

**EXAMPLE**

**Graph Vertical Translations**

State the vertical shift, equation of the midline, amplitude, and period for \( y = \tan \theta - 2 \). Then graph the function.

Since \( \tan \theta - 2 = \tan \theta + (-2) \), \( k = -2 \), and the vertical shift is \(-2\). Draw the midline, \( y = -2 \).

The tangent function has no amplitude and the period is the same as that of \( \tan \theta \).

Draw the graph of the function relative to the midline.
2. State the vertical shift, equation of the midline, amplitude, and period for \( y = \frac{1}{2} \sin \theta + 1 \). Then graph the function.

In general, use the following steps to graph any trigonometric function.

**CONCEPT SUMMARY**

**Graphing Trigonometric Functions**

<table>
<thead>
<tr>
<th>Step</th>
<th>Instructions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 1</td>
<td>Determine the vertical shift, and graph the midline.</td>
</tr>
<tr>
<td>Step 2</td>
<td>Determine the amplitude, if it exists. Use dashed lines to indicate the maximum and minimum values of the function.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Determine the period of the function and graph the appropriate function.</td>
</tr>
<tr>
<td>Step 4</td>
<td>Determine the phase shift and translate the graph accordingly.</td>
</tr>
</tbody>
</table>

**EXAMPLE**

State the vertical shift, amplitude, period, and phase shift of 
\( y = 4 \cos \left[ \frac{1}{2} \left( \theta - \frac{\pi}{3} \right) \right] - 6 \). Then graph the function.

The function is written in the form \( y = a \cos \left[ b(\theta - h) \right] + k \). Identify the values of \( k \), \( a \), \( b \), and \( h \).

- \( k = -6 \), so the vertical shift is \(-6\).
- \( a = 4 \), so the amplitude is \( 4 \) or 4.
- \( b = \frac{1}{2} \), so the period is \( \frac{2\pi}{\frac{1}{2}} \) or \( 4\pi \).
- \( h = \frac{\pi}{3} \), so the phase shift is \( \frac{\pi}{3} \) to the right.

**Step 1** The vertical shift is \(-6\). Graph the midline \( y = -6 \).

**Step 2** The amplitude is 4. Draw dashed lines 4 units above and below the midline at \( y = -2 \) and \( y = -10 \).

**Step 3** The period is \( 4\pi \), so the graph will be stretched.
Graph \( y = 4 \cos \left[ \frac{1}{2} \left( \theta - \frac{\pi}{3} \right) \right] - 6 \) using the midline as a reference.

**Step 4** Shift the graph \( \frac{\pi}{3} \) to the right.

Graph each equation.

3. State the vertical shift, amplitude, period, and phase shift of 
\( y = 3 \sin \left[ \frac{1}{3} \left( \theta - \frac{\pi}{2} \right) \right] + 2 \). Then graph the function.

**Check Your Progress**

**Personal Tutor** at algebra2.com
Suppose a person’s resting blood pressure is 120 over 80. This means that the blood pressure oscillates between a maximum of 120 and a minimum of 80. If this person’s resting heart rate is 60 beats per minute, write a sine function that represents the blood pressure at time \( t \) seconds. Then graph the function.

**Explore**
You know that the function is periodic and can be modeled using sine.

**Plan**
Let \( P \) represent blood pressure and let \( t \) represent time in seconds. Use the equation
\[
P = a \sin \left[ b \left( \frac{t - h}{c} \right) \right] + k.
\]

**Solve**
- Write the equation for the midline. Since the maximum is 120 and the minimum is 80, the midline lies halfway between these values.
  \[
P = \frac{120 + 80}{2} \quad \text{or} \quad 100
\]
- Determine the amplitude by finding the difference between the midline value and the maximum and minimum values.
  \[
a = |120 - 100| \quad \text{or} \quad 20
\]
  \[
a = |80 - 100| \quad \text{or} \quad 20
\]
  Thus, \( a = 20 \).
- Determine the period of the function and solve for \( b \). Recall that the period of a function can be found using the expression \( \frac{2\pi}{|b|} \).
  Since the heart rate is 60 beats per minute, there is one heartbeat, or cycle, per second. So, the period is 1 second.
  \[
1 = \frac{2\pi}{|b|} \quad \text{Write an equation.}
\]
  \[
|b| = 2\pi \quad \text{Multiply each side by } |b|.
\]
  \[
b = \pm 2\pi \quad \text{Solve.}
\]
  For this example, let \( b = 2\pi \). The use of the positive or negative value depends upon whether you begin a cycle with a maximum value (positive) or a minimum value (negative).
  - There is no phase shift, so \( h = 0 \). So, the equation is \( P = 20 \sin 2\pi t + 100 \).
- Graph the function.

**Step 1**
Draw the midline \( P = 100 \).

**Step 2**
Draw maximum and minimum reference lines.

**Step 3**
Use the period to draw the graph of the function.

**Step 4**
There is no phase shift.

**Check**
Notice that each cycle begins at the midline, rises to 120, drops to 80, and then returns to the midline. This represents the blood pressure of 120 over 80 for one heartbeat. Since each cycle lasts 1 second, there will be 60 cycles, or heartbeats, in 1 minute. Therefore, the graph accurately represents the information.
4. Suppose that while doing some moderate physical activity, the person’s blood pressure is 130 over 90 and that the person has a heart rate of 90 beats per minute. Write a sine function that represents the person’s blood pressure at time $t$ seconds. Then graph the function.

**Example 1** (p. 830)
State the amplitude, period, and phase shift for each function. Then graph the function.

1. $y = \sin \left( \theta - \frac{\pi}{2} \right)$
2. $y = \tan (\theta + 60^\circ)$
3. $y = \cos (\theta - 45^\circ)$
4. $y = \sec \left( \theta + \frac{\pi}{3} \right)$

**Example 2** (pp. 831–832)
State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

5. $y = \cos \theta + \frac{1}{4}$
6. $y = \sec \theta - 5$
7. $y = \tan \theta + 4$
8. $y = \sin \theta + 0.25$

**Example 3** (p. 832)
State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

9. $y = 3 \sin \left[ 2(\theta - 30^\circ) \right] + 10$
10. $y = 2 \cot (3\theta + 135^\circ) - 6$
11. $y = \frac{1}{2} \sec \left[ 4\left( \theta - \frac{\pi}{4} \right) \right] + 1$
12. $y = \frac{2}{3} \cos \left[ \frac{1}{2}\left( \theta + \frac{\pi}{6} \right) \right] - 2$

**Example 4** (p. 833)
**PHYSICS** For Exercises 13–15, use the following information.
A weight is attached to a spring and suspended from the ceiling. At equilibrium, the weight is located 4 feet above the floor. The weight is pulled down 1 foot and released.

13. Determine the vertical shift, amplitude, and period of a function that represents the height of the weight above the floor if the weight returns to its lowest position every 4 seconds.

14. Write the equation for the height $h$ of the weight above the floor as a function of time $t$ seconds.

15. Draw a graph of the function you wrote in Exercise 14.

**Exercises**
State the amplitude, period, and phase shift for each function. Then graph the function.

16. $y = \cos (\theta + 90^\circ)$
17. $y = \cot (\theta - 30^\circ)$
18. $y = \sin \left( \theta - \frac{\pi}{4} \right)$
19. $y = \cos \left( \theta + \frac{\pi}{3} \right)$
20. $y = \frac{1}{4} \tan (\theta + 22.5^\circ)$
21. $y = 3 \sin (\theta - 75^\circ)$

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.

22. $y = \sin \theta - 1$
23. $y = \sec \theta + 2$
24. $y = \cos \theta - 5$
25. $y = \csc \theta - \frac{3}{4}$
26. $y = \frac{1}{2} \sin \theta + \frac{1}{2}$
27. $y = 6 \cos \theta + 1.5$
State the vertical shift, amplitude, period, and phase shift for each function. Then graph the function.

28. \( y = 2 \sin \left[ 3(\theta - 45^\circ) \right] + 1 \)
29. \( y = 4 \cos \left[ 2(\theta + 30^\circ) \right] - 5 \)
30. \( y = 3 \csc \left[ \frac{1}{2} (\theta + 60^\circ) \right] - 3.5 \)
31. \( y = 6 \cot \left[ \frac{2}{3} (\theta - 90^\circ) \right] + 0.75 \)
32. \( y = \frac{1}{4} \cos (2\theta - 150^\circ) + 1 \)
33. \( y = \frac{2}{5} \tan (6\theta + 135^\circ) - 4 \)
34. \( y = 3 + 2 \sin \left[ \left( 2\theta + \frac{\pi}{4} \right) \right] \)
35. \( y = 4 + 5 \sec \left[ \frac{1}{3} \left( \theta + \frac{2\pi}{3} \right) \right] \)

**ZOOLOGY** For Exercises 36–38, use the following information.

The population of predators and prey in a closed ecological system tends to vary periodically over time. In a certain system, the population of owls \( O \) can be represented by \( O = 150 + 30 \sin \left( \frac{\pi}{10} t \right) \) where \( t \) is the time in years since January 1, 2001. In that same system, the population of mice \( M \) can be represented by \( M = 600 + 300 \sin \left( \frac{\pi}{10} t + \frac{\pi}{20} \right) \).

36. Find the maximum number of owls. After how many years does this occur?
37. What is the minimum number of mice? How long does it take for the population of mice to reach this level?
38. Why would the maximum owl population follow behind the population of mice?
39. Graph \( y = 3 - \frac{1}{2} \cos \theta \) and \( y = 3 + \frac{1}{2} \cos (\theta + \pi) \). How do the graphs compare?
40. Compare the graphs of \( y = -\sin \left[ \frac{1}{4} \left( \theta - \frac{\pi}{2} \right) \right] \) and \( y = \cos \left[ \frac{1}{4} \left( \theta + \frac{3\pi}{2} \right) \right] \).
41. Graph \( y = 5 + \tan \left( \theta + \frac{\pi}{4} \right) \). Describe the transformation to the parent graph \( y = \tan \theta \).
42. Draw a graph of the function \( y = \frac{2}{3} \cos (\theta - 50^\circ) + 2 \). How does this graph compare to the graph of \( y = \cos \theta \)?

**MUSIC** When represented on oscilloscope, the note A above middle C has a period of \( \frac{1}{440} \). Which of the following can be an equation for an oscilloscope graph of this note? The amplitude of the graph is \( K \).

a. \( y = K \sin 220\pi t \)
   b. \( y = K \sin 440\pi t \)
   c. \( y = K \sin 880\pi t \)

**TIDES** The height of the water in a harbor rose to a maximum height of 15 feet at 6:00 P.M. and then dropped to a minimum level of 3 feet by 3:00 A.M. Assume that the water level can be modeled by the sine function. Write an equation that represents the height \( h \) of the water \( t \) hours after noon on the first day.

**OPEN ENDED** Write the equation of a trigonometric function with a phase shift of \(-45^\circ\). Then graph the function, and its parent graph.

**CHALLENGE** The graph of \( y = \cot \theta \) is a transformation of the graph of \( y = \tan \theta \). Determine \( a, b, \) and \( h \) so that \( \cot \theta = a \tan \left[ b(\theta - h) \right] \) for all values of \( \theta \) for which each function is defined.
Find the amplitude, if it exists, and period of each function. Then graph each function.

\[ 50. \quad y = \csc \theta \]

\[ 51. \quad y = \sin \frac{\theta}{2} \]

\[ 52. \quad y = 3 \tan \frac{2}{3} \theta \]

Solve each equation. Round to the nearest hundredth.

\[ 58. \quad 4^x = 24 \]

\[ 59. \quad 4.3^{3x} + 1 = 78.5 \]

\[ 60. \quad 7^x - 2 = 53 - x \]

Find the total number of diagonals that can be drawn in a decagon.

\[ 62. \quad \frac{w + 12}{4w - 16} - \frac{w + 4}{2w - 8} \]

\[ 63. \quad \frac{3y + 1}{2y - 10} + \frac{1}{y^2 - 2y - 15} \]

Find the value of each function.

\[ 64. \quad \cos 150^\circ \]

\[ 65. \quad \tan 135^\circ \]

\[ 66. \quad \sin \frac{3\pi}{2} \]

\[ 67. \quad \cos \left(-\frac{\pi}{3}\right) \]

\[ 68. \quad \sin (-\pi) \]

\[ 69. \quad \tan \left(-\frac{5\pi}{6}\right) \]

\[ 70. \quad \cos 225^\circ \]

\[ 71. \quad \tan 405^\circ \]
A model for the height of a baseball after it is hit as a function of time can be determined using trigonometry. If the ball is hit with an initial velocity of \( v \) feet per second at an angle of \( \theta \) from the horizontal, then the height \( h \) of the ball after \( t \) seconds can be represented by

\[
h = \left( \frac{-16}{v^2 \cos^2 \theta} \right) t^2 + \left( \frac{\sin \theta}{\cos \theta} \right) t + h_0,
\]

where \( h_0 \) is the height of the ball in feet the moment it is hit.

Find Trigonometric Values \( \) In the equation above, the second term \( \left( \frac{\sin \theta}{\cos \theta} \right) t \) can also be written as \( (\tan \theta) t \). \( (\frac{\sin \theta}{\cos \theta}) t = (\tan \theta) t \) is an example of a trigonometric identity. A **trigonometric identity** is an equation involving trigonometric functions that is true for all values for which every expression in the equation is defined.

The identity \( \tan \theta = \frac{\sin \theta}{\cos \theta} \) is true except for angle measures such as 90°, 270°, 450°, ..., 90° + 180° \( \cdot \) \( k \). The cosine of each of these angle measures is 0, so none of the expressions \( \tan 90^\circ \), \( \tan 270^\circ \), \( \tan 450^\circ \), and so on, are defined. An identity similar to this is \( \cot \theta = \frac{\cos \theta}{\sin \theta} \).

These identities are sometimes called **quotient identities**. These and other basic trigonometric identities are listed below.

<table>
<thead>
<tr>
<th>KEY CONCEPT</th>
<th>Basic Trigonometric Identities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Quotient Identities</strong></td>
<td>( \tan \theta = \frac{\sin \theta}{\cos \theta} ), ( \cos \theta \neq 0 ) \hspace{1cm} ( \cot \theta = \frac{\cos \theta}{\sin \theta} ), ( \sin \theta \neq 0 )</td>
</tr>
<tr>
<td><strong>Reciprocal Identities</strong></td>
<td>( \csc \theta = \frac{1}{\sin \theta} \hspace{1cm} \text{sin} \theta \neq 0 ) \hspace{1cm} ( \sec \theta = \frac{1}{\cos \theta} \hspace{1cm} \text{cos} \theta \neq 0 ) \hspace{1cm} ( \cot \theta = \frac{1}{\tan \theta} \hspace{1cm} \text{tan} \theta \neq 0 )</td>
</tr>
<tr>
<td><strong>Pythagorean Identities</strong></td>
<td>( \cos^2 \theta + \sin^2 \theta = 1 ) \hspace{1cm} ( \tan^2 \theta + 1 = \sec^2 \theta ) \hspace{1cm} ( \cot^2 \theta + 1 = \csc^2 \theta )</td>
</tr>
</tbody>
</table>

You can use trigonometric identities to find values of trigonometric functions.
It is often easiest to write all expressions in terms of sine and/or cosine.

**EXAMPLE** Find a Value of a Trigonometric Function

**a.** Find \( \cos \theta \) if \( \sin \theta = \frac{-3}{5} \) and \( 90^\circ < \theta < 180^\circ \).

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 & \text{Trigonometric identity} \\
\cos^2 \theta &= 1 - \sin^2 \theta & \text{Subtract } \sin^2 \theta \text{ from each side.} \\
\cos^2 \theta &= 1 - \left( \frac{3}{5} \right)^2 & \text{Substitute } \frac{3}{5} \text{ for } \sin \theta. \\
\cos^2 \theta &= 1 - \frac{9}{25} & \text{Square } \frac{3}{5}. \\
\cos^2 \theta &= \frac{16}{25} & \text{Subtract.} \\
\cos \theta &= \pm \frac{4}{5} & \text{Take the square root of each side.}
\end{align*}
\]

Since \( \theta \) is in the second quadrant, \( \cos \theta \) is negative.

Thus, \( \cos \theta = -\frac{4}{5} \).

**b.** Find \( \csc \theta \) if \( \cot \theta = -\frac{1}{4} \) and \( 270^\circ < \theta < 360^\circ \).

\[
\begin{align*}
\cot^2 \theta + 1 &= \csc^2 \theta & \text{Trigonometric identity} \\
\left(-\frac{1}{4}\right)^2 + 1 &= \csc^2 \theta & \text{Substitute } -\frac{1}{4} \text{ for } \cot \theta. \\
\frac{1}{16} + 1 &= \csc^2 \theta & \text{Square } -\frac{1}{4}. \\
\frac{17}{16} &= \csc^2 \theta & \text{Add.} \\
\pm \frac{\sqrt{17}}{4} &= \csc \theta & \text{Take the square root of each side.}
\end{align*}
\]

Since \( \theta \) is in the fourth quadrant, \( \csc \theta \) is negative.

Thus, \( \csc \theta = -\frac{\sqrt{17}}{4} \).

**CHECK Your Progress**

1A. Find \( \sin \theta \) if \( \cos \theta = \frac{1}{3} \) and \( 270^\circ < \theta < 360^\circ \).

1B. Find \( \sec \theta \) if \( \sin \theta = -\frac{2}{7} \) and \( 180^\circ < \theta < 270^\circ \).

**SIMPLIFY EXPRESSIONS** Trigonometric identities can also be used to simplify expressions containing trigonometric functions. Simplifying an expression that contains trigonometric functions means that the expression is written as a numerical value or in terms of a single trigonometric function, if possible.

**EXAMPLE** Simplify an Expression

Simplify \( \frac{\csc^2 \theta - \cot^2 \theta}{\cos \theta} \).

\[
\begin{align*}
\frac{\csc^2 \theta - \cot^2 \theta}{\cos \theta} &= \frac{\frac{1}{\sin^2 \theta} - \frac{\cos^2 \theta}{\sin^2 \theta}}{\cos \theta} & \csc^2 \theta = \frac{1}{\sin^2 \theta} , \cot^2 \theta = \frac{\cos^2 \theta}{\sin^2 \theta} \\
&= \frac{1 - \cos^2 \theta}{\sin^2 \theta} \cdot \frac{\sin^2 \theta}{\cos \theta} & \text{Add.} \\
&= \frac{\sin^2 \theta}{\cos \theta} \cdot \frac{\sin^2 \theta}{\cos \theta}
\end{align*}
\]
\[
\begin{align*}
\sin^2 \theta &= \frac{\sin^2 \theta}{\cos \theta} \quad 1 - \cos^2 \theta = \sin^2 \theta \\
= \frac{1}{\cos \theta} &\quad \frac{\sin^2 \theta}{\sin^2 \theta} = 1 \\
= \sec \theta &\quad \frac{1}{\cos \theta} = \sec \theta
\end{align*}
\]

Simplify each expression.

2A. \(\frac{\tan^2 \theta \csc^2 \theta - 1}{\sec^2 \theta}\)  
2B. \(\frac{\sec \theta}{\sin \theta (1 - \cos^2 \theta)}\)

**Example**

**Simplify and Use an Expression**

**BASEBALL** Refer to the application at the beginning of the lesson. Rewrite the equation in terms of \(\tan \theta\).

\[
h = \left(\frac{-16}{v^2 \cos^2 \theta}\right)t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \quad \text{Original equation}
\]

\[
= \frac{-16}{v^2}(\frac{1}{\cos^2 \theta})t^2 + \left(\frac{\sin \theta}{\cos \theta}\right)t + h_0 \quad \text{Factor.}
\]

\[
= \frac{-16}{v^2}(\sec^2 \theta)t^2 + (\tan \theta)t + h_0 \quad \sin \frac{\theta}{\cos \theta} = \tan \theta
\]

\[
= \frac{-16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0 \quad \text{Since } \frac{1}{\cos \theta} = \sec \theta, \frac{1}{\cos^2 \theta} = \sec^2 \theta.
\]

\[
= \frac{-16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0 \quad \sec^2 \theta = 1 + \tan^2 \theta
\]

Thus, \(\frac{-16}{v^2 \cos^2 \theta}\) + \(\frac{\sin \theta}{\cos \theta}\) + \(h_0 = \frac{-16}{v^2}(1 + \tan^2 \theta)t^2 + (\tan \theta)t + h_0\).

**Check Your Progress**

3. Rewrite the expression \(\cot^2 \theta - \tan^2 \theta\) in terms of \(\sin \theta\).

**Example 1**

Find the value of each expression.

1. \(\tan \theta\), if \(\sin \theta = \frac{4}{5}\); \(90^\circ \leq \theta < 180^\circ\)
2. \(\csc \theta\), if \(\cos \theta = -\frac{3}{5}\); \(180^\circ \leq \theta < 270^\circ\)
3. \(\cos \theta\), if \(\sin \theta = \frac{4}{5}\); \(0^\circ \leq \theta < 90^\circ\)
4. \(\sec \theta\), if \(\tan \theta = -1\); \(270^\circ < \theta < 360^\circ\)

**Example 2**

Simplify each expression.

5. \(\csc \theta \cos \theta \tan \theta\)
6. \(\sec^2 \theta - 1\)
7. \(\frac{\tan \theta}{\sin \theta}\)
8. \(\sin \theta (1 + \cot^2 \theta)\)

**Example 3**

9. **PHYSICAL SCIENCE** When a person moves along a circular path, the body leans away from a vertical position. The nonnegative acute angle that the body makes with the vertical is called the angle of inclination and is represented by the equation \(\tan \theta = \frac{v^2}{gR}\), where \(R\) is the radius of the circular path, \(v\) is the speed of the person in meters per second, and \(g\) is the acceleration due to gravity, 9.8 meters per second squared. Write an equivalent expression using \(\sin \theta\) and \(\cos \theta\).
Find the value of each expression.

10. \( \tan \theta \), if \( \cot \theta = 2; \) \( 0^\circ \leq \theta < 90^\circ \)
11. \( \sin \theta \), if \( \cos \theta = \frac{2}{3}; \) \( 0^\circ \leq \theta < 90^\circ \)
12. \( \sec \theta \), if \( \tan \theta = -2; \) \( 90^\circ < \theta < 180^\circ \)
13. \( \tan \theta \), if \( \sec \theta = -3; \) \( 180^\circ < \theta < 270^\circ \)
14. \( \csc \theta \), if \( \cos \theta = -\frac{3}{5}; \) \( 90^\circ < \theta < 180^\circ \)
15. \( \cos \theta \), if \( \sec \theta = \frac{5}{3}; \) \( 270^\circ < \theta < 360^\circ \)
16. \( \cos \theta \), if \( \sin \theta = \frac{1}{2}; \) \( 0^\circ \leq \theta < 90^\circ \)
17. \( \csc \theta \), if \( \cos \theta = -\frac{2}{3}; \) \( 180^\circ < \theta < 270^\circ \)

Simplify each expression.

18. \( \cos \theta \csc \theta \)
19. \( \tan \theta \cot \theta \)
20. \( \sin \theta \cot \theta \)
21. \( \cos \theta \tan \theta \)
22. \( 2(\csc^2 \theta - \cot^2 \theta) \)
23. \( 3(\tan^2 \theta - \sec^2 \theta) \)
24. \( \cos \csc \theta \tan \theta \)
25. \( \sin \theta \csc \theta \frac{\cot \theta}{\cos \theta} \)
26. \( \frac{1 - \cos^2 \theta}{\sin^2 \theta} \)

**ELECTRONICS** For Exercises 27 and 28, use the following information.

When an alternating current of frequency \( f \) and a peak current \( I \) pass through a resistance \( R \), then the power delivered to the resistance at time \( t \) seconds is \( P = I^2R - I^2R \cos^2 2ft \pi \).

27. Write an expression for the power in terms of \( \sin^2 2ft \pi \).
28. Write an expression for the power in terms of \( \tan^2 2ft \pi \).

Find the value of each expression.

29. \( \tan \theta \), if \( \cos \theta = \frac{4}{5}; \) \( 0^\circ \leq \theta < 90^\circ \)
30. \( \cos \theta \), if \( \csc \theta = -\frac{5}{3}; \) \( 270^\circ < \theta < 360^\circ \)
31. \( \sec \theta \), if \( \sin \theta = \frac{3}{4}; \) \( 90^\circ < \theta < 180^\circ \)
32. \( \sin \theta \), if \( \tan \theta = 4; \) \( 180^\circ < \theta < 270^\circ \)

Simplify each expression.

33. \( \frac{1 - \sin^2 \theta}{\sin^2 \theta} \)
34. \( \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta} \)
35. \( \frac{\tan^2 \theta - \sin^2 \theta}{\tan^2 \theta \sin^2 \theta} \)

**AMUSEMENT PARKS** For Exercises 36–38, use the following information.

Suppose a child is riding on a merry-go-round and is seated on an outside horse. The diameter of the merry-go-round is 16 meters.

36. Refer to Exercise 9. If the sine of the angle of inclination of the child is \( \frac{1}{5} \), what is the angle of inclination made by the child?
37. What is the velocity of the merry-go-round?
38. If the speed of the merry-go-round is 3.6 meters per second, what is the value of the angle of inclination of a rider?

**LIGHTING** For Exercises 39 and 40, use the following information.

The amount of light that a source provides to a surface is called the illuminance. The illuminance \( E \) in foot candles on a surface is related to the distance \( R \) in feet from the light source. The formula \( \sec \theta = \frac{1}{ER^2} \), where \( I \) is the intensity of the light source measured in candles and \( \theta \) is the angle between the light beam and a line perpendicular to the surface, can be used in situations in which lighting is important.

39. Solve the formula in terms of \( E \).
40. Is the equation in Exercise 39 equivalent to \( R^2 = \frac{I \tan \theta \cos \theta}{E} \)? Explain.
41. **REASONING** Describe how you can determine the quadrant in which the terminal side of angle \( \alpha \) lies if \( \sin \alpha = -\frac{1}{4} \).

42. **OPEN ENDED** Write two expressions that are equivalent to \( \tan \theta \sin \theta \).

43. **REASONING** If \( \cot(x) = \cot \left( \frac{\pi}{3} \right) \) and \( 3\pi < x < 4\pi \), find \( x \).

44. **CHALLENGE** If \( \tan \beta = \frac{3}{4} \), find \( \frac{\sin \beta \sec \beta}{\cot \beta} \).

45. **Writing in Math** Use the information on page 837 to explain how trigonometry can be used to model the path of a baseball. Include an explanation of why the equation at the beginning of the lesson is the same as \( y = -\frac{16 \sec^2 \theta}{v^2} x^2 + (\tan \theta) x + h_0 \).

### STANDARDIZED TEST PRACTICE

#### ACT/SAT

If \( \sin x = m \) and \( 0 < x < 90^\circ \), then \( \tan x = \)  
- **A** \( \frac{1}{m^2} \)  
- **B** \( \frac{1 - m^2}{m} \)  
- **C** \( \frac{m}{\sqrt{1 - m^2}} \)  
- **D** \( \frac{m}{1 - m^2} \)

#### REVIEW

Refer to the figure below. If \( \cos D = 0.8 \), what is length \( DF \)?

<table>
<thead>
<tr>
<th>Option</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>5</td>
</tr>
<tr>
<td>G</td>
<td>4</td>
</tr>
<tr>
<td>H</td>
<td>3.2</td>
</tr>
<tr>
<td>J</td>
<td>( \frac{4}{5} )</td>
</tr>
</tbody>
</table>

State the vertical shift, equation of the midline, amplitude, and period for each function. Then graph the function.  

**Lesson 14-2**

48. \( y = \sin \theta - 1 \)

49. \( y = \tan \theta + 12 \)

Find the amplitude, if it exists, and period of each function. Then graph each function.  

**Lesson 14-1**

50. \( y = \csc 2\theta \)

51. \( y = \cos 3\theta \)

52. \( y = \frac{1}{3} \cot 5\theta \)

53. Find the sum of a geometric series for which \( a_1 = 48, a_n = 3 \), and \( r = \frac{1}{2} \).  

54. Write an equation of a parabola with focus at \((11, -1)\) and directrix \( y = 2 \).  

**Lesson 10-2**

55. **TEACHING** Ms. Granger has taught 288 students at this point in her career. If she has 30 students each year from now on, the function \( S(t) = 288 + 30t \) gives the number of students \( S(t) \) she will have taught after \( t \) more years.  

How many students will she have taught after 7 more years?  

**Lesson 2-1**

56. If \( 4 + 8 = 12 \), then \( 12 = 4 + 8 \).

57. If \( 7 + s = 21 \), then \( s = 14 \).

58. If \( 4x = 16 \), then \( 12x = 48 \).

59. If \( q + (8 + 5) = 32 \), then \( q + 13 = 32 \).
**Main Ideas**
- Verify trigonometric identities by transforming one side of an equation into the form of the other side.
- Verify trigonometric identities by transforming each side of the equation into the same form.

*GET READY for the Lesson*

Examine the graphs at the right. Recall that when the graphs of two functions coincide, the functions are equivalent. However, the graphs only show a limited range of solutions. It is not sufficient to show some values of \( \theta \) and conclude that the statement is true for all values of \( \theta \). In order to show that the equation \( \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta \) for all values of \( \theta \), you must consider the general case.

**Transform One Side of an Equation** You can use the basic trigonometric identities along with the definitions of the trigonometric functions to verify identities. For example, if you wish to show that \( \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta \) is an identity, you need to show that it is true for all values of \( \Theta \).

Verifying an identity is like checking the solution of an equation. You must simplify one or both sides of an equation separately until they are the same. In many cases, it is easier to work with only one side of an equation. You may choose either side, but it is often easier to begin with the more complicated side of the equation. Transform that expression into the form of the simpler side.

**EXAMPLE** 

**Transform One Side of an Equation**

Verify that \( \tan^2 \theta - \sin^2 \theta = \tan^2 \theta \sin^2 \theta \) is an identity.

Transform the left side.

\[
\begin{align*}
\tan^2 \theta - \sin^2 \theta &= \tan^2 \theta \sin^2 \theta & \text{Original equation} \\
= \frac{\sin^2 \theta}{\cos^2 \theta} - \sin^2 \theta &= \tan^2 \theta \sin^2 \theta & \text{since} \quad \tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta} \\
= & \frac{\sin^2 \theta \cos^2 \theta - \sin^2 \theta \cos^2 \theta}{\cos^2 \theta} &= \tan^2 \theta \sin^2 \theta & \text{Rewrite using the LCD, } \cos^2 \theta. \\
= & \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} = \tan^2 \theta \sin^2 \theta & \text{Subtract.} \\
= & \frac{\sin^2 \theta (1 - \cos^2 \theta)}{\cos^2 \theta} = \tan^2 \theta \sin^2 \theta & \text{Factor.}
\end{align*}
\]
1. Verify that \( \cot^2 \theta - \cos^2 \theta = \cot^2 \theta \cos^2 \theta \) is an identity.

\[
\frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta \sin \theta \\
\frac{\sin^2 \theta \cdot \sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta \sin^2 \theta \\
\tan^2 \theta \sin^2 \theta = \tan^2 \theta \sin^2 \theta
\]

\[
1 - \cos^2 \theta = \sin^2 \theta
\]

\[
\frac{ab}{c} = \frac{a}{c} \cdot \frac{b}{1}
\]

\[
\frac{\sin^2 \theta}{\cos^2 \theta} = \tan \theta
\]

2. Find an equivalent expression

\[
\sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cot \theta} \right) =
\]

A \( \cos \theta \)  
B \( \sin \theta \)  
C \( \cos^2 \theta \)  
D \( \sin^2 \theta \)

**Read the Test Item**

Find an expression that is equal to the given expression.

**Solve the Test Item**

Transform the given expression to match one of the choices.

\[
\sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cot \theta} \right) = \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta}{\cos \theta \sin \theta} \right)
\]

\[
= \sin \theta \left( \frac{1}{\sin \theta} - \frac{\cos \theta \sin \theta}{\cos \theta} \right)
\]

Simplify.

\[
= \sin \theta \left( \frac{1}{\sin \theta} - \sin \theta \right)
\]

Simplify.

\[
= 1 - \sin^2 \theta
\]

Distributive Property

\[
= \cos^2 \theta
\]

The answer is C.

**Transform Both Sides of an Equation**

Sometimes it is easier to transform both sides of an equation separately into a common form. The following suggestions may be helpful as you verify trigonometric identities.

- Substitute one or more basic trigonometric identities to simplify an expression.
- Factor or multiply to simplify an expression.
- Multiply both the numerator and denominator by the same trigonometric expression.
- Write both sides of the identity in terms of sine and cosine only. Then simplify each side as much as possible.
EXAMPLE

Verify by Transforming Both Sides

Verify that $\sec^2 \theta - \tan^2 \theta = \tan \theta \cot \theta$ is an identity.

\[
\begin{align*}
\sec^2 \theta - \tan^2 \theta &\not\triangleq \tan \theta \cot \theta & \text{Original equation} \\
\frac{1}{\cos^2 \theta} - \frac{\sin^2 \theta}{\cos^2 \theta} &\not\triangleq \frac{\sin \theta}{\cos \theta} \cdot \frac{\cos \theta}{\sin \theta} \\
1 - \frac{\sin^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
\frac{\cos^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
1 &\not\triangleq 1 \\
\end{align*}
\]

Express all terms using sine and cosine.

\[
\begin{align*}
1 - \frac{\sin^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
\frac{\cos^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
1 &\not\triangleq 1 \\
\end{align*}
\]

Subtract on the left. Multiply on the right.

\[
\begin{align*}
\frac{\cos^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
1 &\not\triangleq 1 \\
\frac{\sin^2 \theta}{\cos^2 \theta} &\not\triangleq 1 \\
\end{align*}
\]

Simplify the left side.

3. Verify that $\csc^2 \theta - \cot^2 \theta = \cot \theta \tan \theta$ is an identity.

### Check Your Progress

Verify that each of the following is an identity.

1. $\tan \theta (\cot \theta + \tan \theta) = \sec^2 \theta$
2. $\tan^2 \theta \cos^2 \theta = 1 - \cos^2 \theta$
3. $\frac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$
4. $\frac{1 + \tan^2 \theta}{\csc^2 \theta} = \tan^2 \theta$
5. $\sin \theta = \frac{1}{\tan \theta + \cot \theta}$
6. $\frac{\sec \theta + 1}{\tan \theta} = \frac{\tan \theta}{\sec \theta - 1}$

### STANDARDIZED TEST PRACTICE

Which expression can be used to form an identity with $\frac{\sec \theta + \csc \theta}{1 + \tan \theta}$?

A $\sin \theta$  B $\cos \theta$  C $\tan \theta$  D $\csc \theta$

### Exercises

Verify that each of the following is an identity.

8. $\cos^2 \theta + \tan^2 \theta \cos^2 \theta = 1$
9. $\cot \theta (\cot \theta + \tan^2 \theta) = \csc^2 \theta$
10. $1 + \sec^2 \theta \sin^2 \theta = \sec^2 \theta$
11. $\sin \theta \sec \theta \cot \theta = 1$
12. $\frac{1 - \cos \theta}{1 + \cos \theta} = (\csc \theta - \cot \theta)^2$
13. $\frac{1 - 2 \cos^2 \theta}{\sin \theta \cos \theta} = \tan \theta - \cot \theta$
14. $\cot \theta \csc \theta = \frac{\cot \theta + \csc \theta}{\sin \theta + \tan \theta}$
15. $\sin \theta + \cos \theta = \frac{1 + \tan \theta}{\sec \theta}$
16. $\frac{\sec \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = \cot \theta$
17. $\frac{\sin \theta}{1 - \cos \theta} + \frac{1 - \cos \theta}{\sin \theta} = 2 \csc \theta$
18. Verify that $\tan \theta \sin \theta \cos \theta \csc^2 \theta = 1$ is an identity.
19. Show that $1 + \cos \theta$ and $\frac{\sin^2 \theta}{1 - \cos \theta}$ form an identity.
**PHYSICS** For Exercises 20 and 21, use the following information.

If an object is propelled from ground level, the maximum height that it reaches is given by \( h = \frac{v^2 \sin^2 \theta}{2g} \), where \( \theta \) is the angle between the ground and the initial path of the object, \( v \) is the object’s initial velocity, and \( g \) is the acceleration due to gravity, 9.8 meters per second squared.

20. Verify the identity \( \frac{v^2 \sin^2 \theta}{2g} = \frac{v^2 \tan^2 \theta}{2g \sec^2 \theta} \).

21. A model rocket is launched with an initial velocity of 110 meters per second at an angle of 80˚ with the ground. Find the maximum height of the rocket.

598.7 m

Verify that each of the following is an identity.

22. \( \frac{1 + \sin \theta}{\sin \theta} = \frac{\cot^2 \theta}{\csc \theta - 1} \)

23. \( \frac{1 + \tan \theta}{1 + \cot \theta} = \frac{\sin \theta}{\cos \theta} \)

24. \( \frac{1}{\sec^2 \theta} + \frac{1}{\csc^2 \theta} = 1 \)

25. \( 1 + \frac{1}{\cos \theta} = \frac{\tan^2 \theta}{\sec \theta - 1} \)

26. \( 1 - \tan^4 \theta = 2 \sec^2 \theta - \sec^4 \theta \)

27. \( \cos^4 \theta - \sin^4 \theta = \cos^2 \theta - \sin^2 \theta \)

28. \( \frac{1 - \cos \theta}{\sin \theta} = \frac{\sin \theta}{1 + \cos \theta} \)

VERIFYING TRIGONOMETRIC IDENTITIES You can determine whether or not an equation may be a trigonometric identity by graphing the expressions on either side of the equals sign as two separate functions. If the graphs do not match, then the equation is not an identity. If the two graphs do coincide, the equation might be an identity. The equation has to be verified algebraically to ensure that it is an identity.

Determine whether each of the following may be or is not an identity.

30. \( \cot x + \tan x = \csc x \cot x \)

31. \( \sec^2 x - 1 = \sin^2 x \sec^2 x \)

32. \( (1 + \sin x)(1 - \sin x) = \cos^2 x \)

33. \( \frac{1}{\sec x \tan x} = \csc x - \sin x \)

34. \( \frac{\sec^2 x}{\tan x} = \sec x \csc x \)

35. \( \frac{1}{\sec x} + \frac{1}{\csc x} = 1 \)

36. OPEN END Write a trigonometric equation that is not an identity. Explain how you know it is not an identity.

37. Which One Doesn’t Belong? Identify the equation that does not belong with the other three. Explain your reasoning.

\( \sin^2 \theta + \cos^2 \theta = 1 \quad 1 + \cot^2 \theta = \csc^2 \theta \)

\( \sin^2 \theta - \cos^2 \theta = 2 \sin^2 \theta \quad \tan^2 \theta + 1 = \sec^2 \theta \)

38. CHALLENGE Present a logical argument for why the identity \( \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \) is true when \( 0 \leq x \leq 1 \).

39. Writing in Math Use the information on pages 842 and 843 to explain why you cannot perform operations to each side of an unverified identity and explain why you cannot use the graphs of two expressions to verify an identity.
Find the value of each expression. (Lesson 14-3)

42. sec θ, if tan θ = \( \frac{1}{2} \); 0° < θ < 90°
43. cos θ, if sin θ = \( \frac{-2}{3} \); 180° < θ < 270°
44. csc θ, if cot θ = \( \frac{-7}{12} \); 90° < θ < 180°
45. sin θ, if cos θ = \( \frac{3}{4} \); 270° < θ < 360°

State the amplitude, period, and phase shift of each function. Then graph each function. (Lesson 14-2)

46. \( y = \cos (\theta - 30°) \)
47. \( y = \sin (\theta - 45°) \)
48. \( y = 3 \cos (\theta + \frac{\pi}{2}) \)

49. COMMUNICATIONS The carrier wave for a certain FM radio station can be modeled by the equation \( y = A \sin (10^7 \cdot 2\pi t) \), where \( A \) is the amplitude of the wave and \( t \) is the time in seconds. Determine the period of the carrier wave. (Lesson 14-1)

50. BUSINESS A company estimates that it costs \( 0.03x^2 + 4x + 1000 \) dollars to produce \( x \) units of a product. Find an expression for the average cost per unit. (Lesson 6-3)

Use the related graph of each equation to determine its solutions. (Lesson 5-2)

51. \( y = x^2 + 6x + 5 \)
52. \( y = -3x^2 \)
53. \( y = x^2 + 4x - 4 \)

PREREQUISITE SKILL Simplify each expression. (Lessons 7-5)

54. \( \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \)
55. \( \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \)
56. \( \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2} \)
57. \( \frac{1}{2} - \frac{\sqrt{3}}{4} \)
1. Find the amplitude and period of \( y = \frac{3}{4} \sin \frac{1}{2} \theta \). Then graph the function. (Lesson 14-1)

**POPULATION** For Exercises 2–4 use the following information.
The population of a certain species of deer can be modeled by the function \( p = 30,000 + 20,000 \cos \left( \frac{\pi}{10} t \right) \), where \( p \) is the population and \( t \) is the time in years. (Lesson 14-1)

2. What is the amplitude of the population and what does it represent?
3. What is the period of the function and what does it represent?
4. Graph the function.

5. **MUTLIPLICATION** Find the amplitude, if it exists, and period of \( y = 3 \cot \left( \frac{1}{4} \theta \right) \). (Lesson 14-1)
   
   A. \( 3; \frac{\pi}{4} \)  
   B. \( 3; 4\pi \)  
   C. not defined; \( 4\pi \)  
   D. not defined; \( \frac{\pi}{4} \)

For Exercises 6–9, consider the function \( y = 2 \cos \left[ \frac{1}{4} \left( \theta - \frac{\pi}{4} \right) \right] - 5 \). (Lesson 14-2)

6. State the vertical shift.
7. State the amplitude and period.
8. State the phase shift.
9. Graph the function.

10. **PENDULUM** The position of the pendulum on a particular clock can be modeled using a sine equation. The period of the pendulum is 2 seconds and the phase shift is 0.5 second. The pendulum swings 6 inches to either side of the center position. Write an equation to represent the position of the pendulum \( p \) at time \( t \) seconds. Assume that the \( x \)-axis represents the center line of the pendulum’s path, that the area above the \( x \)-axis represents a swing to the right, and that the pendulum swings to the right first. (Lesson 14-2)

Find the value of each expression. (Lesson 14-3)

11. \( \cos \theta \), if \( \sin \theta = \frac{4}{5}; 90^\circ < \theta < 180^\circ \)

12. \( \csc \theta \), if \( \cot \theta = -\frac{2}{3}; 270^\circ < \theta < 360^\circ \)

13. \( \sec \theta \), if \( \tan \theta = \frac{1}{2}; 0^\circ < \theta < 90^\circ \)

14. **SWINGS** Amy takes her cousin to the park to swing while she is babysitting. The horizontal force that Amy uses to push her cousin can be found using the formula \( F = Mg \tan \theta \), where \( F \) is the force, \( M \) is the mass of the child, \( g \) is gravity, and \( \theta \) is the angle that the swing makes with its resting position. Write an equivalent expressing using \( \sin \theta \) and \( \sec \theta \). (Lesson 14-3)

15. **MULTIPLE CHOICE** Which of the following is equivalent to \( \frac{1 - \sin^2 \theta}{1 - \cos^2 \theta} \cdot \tan \theta \)? (Lesson 14-3)

   F. \( \tan \theta \)  
   G. \( \cot \theta \)  
   H. \( \sin \theta \)  
   J. \( \cos \theta \)

Verify that each of the following is an identity. (Lesson 14-4)

16. \( \tan^2 \theta + 1 = \frac{\tan \theta}{\cos \theta \cdot \sin \theta} \)

17. \( \frac{\sin \theta \cdot \sec \theta}{\sec \theta - 1} = (\sec \theta + 1) \cot \theta \)

18. \( \sin^2 \theta \cdot \tan^2 \theta = \tan^2 \theta - \sin^2 \theta \)

19. \( \cot \theta(1 - \cos \theta) = \frac{\cos \theta \cdot \sin \theta}{1 + \cos \theta} \)

20. **OPTICS** If two prisms of the same power are placed next to each other, their total power can be determined using the formula \( z = 2p \cos \theta \) where \( z \) is the combined power of the prisms, \( p \) is the power of the individual prisms, and \( \theta \) is the angle between the two prisms. Verify the identity \( 2p \cos \theta = 2p(1 - \sin^2 \theta) \sec \theta \). (Lesson 14-4)
Have you ever been talking on a cell phone and temporarily lost the signal? Radio waves that pass through the same place at the same time cause interference. Constructive interference occurs when two waves combine to have a greater amplitude than either of the component waves. Destructive interference occurs when the component waves combine to have a smaller amplitude.

**Sum and Difference Formulas** Notice that the third equation shown above involves the sum of \( \alpha \) and \( \beta \). It is often helpful to use formulas for the trigonometric values of the difference or sum of two angles. For example, you could find \( \sin 15^\circ \) by evaluating \( \sin (60^\circ - 45^\circ) \). Formulas can be developed that can be used to evaluate expressions like \( \sin (\alpha - \beta) \) or \( \cos (\alpha + \beta) \).

The figure at the right shows two angles \( \alpha \) and \( \beta \) in standard position on the unit circle. Use the Distance Formula to find \( d \), where \((x_1, y_1) = (\cos \beta, \sin \beta)\) and \((x_2, y_2) = (\cos \alpha, \sin \alpha)\).

\[
\begin{align*}
    d & = \sqrt{\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2} \\
    d^2 & = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2 \\
    & = (\cos^2 \alpha - 2\cos \alpha \cos \beta + \cos^2 \beta) + (\sin^2 \alpha - 2\sin \alpha \sin \beta + \sin^2 \beta) \\
    & = \cos^2 \alpha + \sin^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \\
    & = 1 + 1 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta \\
    & = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta
\end{align*}
\]

Now find the value of \( d^2 \) when the angle having measure \( \alpha - \beta \) is in standard position on the unit circle, as shown in the figure at the left.

\[
\begin{align*}
    d & = \sqrt{\cos (\alpha - \beta) - 1)^2 + [\sin (\alpha - \beta) - 0]^2} \\
    d^2 & = [\cos (\alpha - \beta) - 1]^2 + [\sin (\alpha - \beta) - 0]^2 \\
    & = [\cos^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1] + \sin^2 (\alpha - \beta) \\
    & = \cos^2 (\alpha - \beta) + \sin^2 (\alpha - \beta) - 2 \cos (\alpha - \beta) + 1 \\
    & = 1 - 2 \cos (\alpha - \beta) + 1 \\
    & = 2 - 2 \cos (\alpha - \beta)
\end{align*}
\]
By equating the two expressions for \( d^2 \), you can find a formula for \( \cos(\alpha - \beta) \).

\[
d^2 = d^2
\]

\[
2 - 2 \cos(\alpha - \beta) = 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta
\]

\[
-1 + \cos(\alpha - \beta) = -1 + \cos \alpha \cos \beta + \sin \alpha \sin \beta
\]

Divide each side by -2.

Add 1 to each side.

Use the formula for \( \cos(\alpha - \beta) \) to find a formula for \( \cos(\alpha + \beta) \).

\[
\cos(\alpha - \beta) = \cos[\alpha - (-\beta)]
\]

\[
= \cos \alpha \cos (-\beta) + \sin \alpha \sin (-\beta)
\]

\[
= \cos \alpha \cos \beta - \sin \alpha \sin \beta
\]

\[
\cos(-\beta) = \cos \beta; \sin(-\beta) = -\sin \beta
\]

You can use a similar method to find formulas for \( \sin(\alpha + \beta) \) and \( \sin(\alpha - \beta) \).

**KEY CONCEPT**

**SUM AND DIFFERENCE OF ANGLES FORMULAS**

The following identities hold true for all values of \( \alpha \) and \( \beta \).

\[
\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta
\]

\[
\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta
\]

Notice the symbol \( \mp \) in the formula for \( \cos(\alpha \pm \beta) \). It means “minus or plus.” In the cosine formula, when the sign on the left side of the equation is plus, the sign on the right side is minus; when the sign on the left side is minus, the sign on the right side is plus. The signs match each other in the sine formula.

**EXAMPLE**

**USE SUM AND DIFFERENCE OF ANGLES FORMULAS**

Find the exact value of each expression.

**a.** \( \cos 75^\circ \)

Use the formula \( \cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta \).

\[
\cos 75^\circ = \cos(30^\circ + 45^\circ)
\]

\[
\alpha = 30^\circ, \beta = 45^\circ
\]

\[
= \cos 30^\circ \cos 45^\circ - \sin 30^\circ \sin 45^\circ
\]

\[
= \left( \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} \right) \quad \left( \frac{1}{2} \cdot \frac{\sqrt{2}}{2} \right)
\]

Evaluate each expression.

\[
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

Multiply.

\[
= \frac{\sqrt{6} - \sqrt{2}}{4}
\]

Simplify.

**b.** \( \sin(-210^\circ) \)

Use the formula \( \sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta \).

\[
\sin(-210^\circ) = \sin(60^\circ - 270^\circ)
\]

\[
\alpha = 60^\circ, \beta = 270^\circ
\]

\[
= \sin 60^\circ \cos 270^\circ - \cos 60^\circ \sin 270^\circ
\]

\[
= \left( \frac{\sqrt{3}}{2} \right)(0) - \left( \frac{1}{2} \right)(-1)
\]

Evaluate each expression.

\[
= 0 - \left( \frac{-1}{2} \right) = \frac{1}{2}
\]

Simplify.
On June 22, the maximum amount of light energy falling on a square foot of ground at a location in the northern hemisphere is given by

\[ E \sin \left( \frac{113.5^\circ - \phi}{H11002} \right) \]

where \( \phi \) is the latitude of the location and \( E \) is the amount of light energy when the Sun is directly overhead. Use the difference of angles formula to determine the amount of light energy in Rochester, New York, located at a latitude of 43.1° N.

Use the difference formula for sine.

\[
\sin \left( 113.5^\circ - \phi \right) / H11002 \sin 113.5^\circ / H11368 \cos \phi / H11002 \cos 113.5^\circ / H11002 \sin \phi \]

\[ = 0.9171 \cdot 0.7302 - (-0.3987) \cdot 0.6833 \]

\[ = 0.9420 \]

In Rochester, New York, the maximum light energy per square foot is 0.9420 \( E \).

2. Determine the amount of light energy in West Hollywood, California, which is located at a latitude of 34.1° N.

Verify Identities You can also use the sum and difference formulas to verify identities.

**EXAMPLE 2** Verify Identities

Verify that each of the following is an identity.

\[ \text{a. } \sin (180^\circ + \theta) = -\sin \theta \]

\[ \sin (180^\circ + \theta) \overset{?}{=} -\sin \theta \quad \text{Original equation} \]

\[ \sin 180^\circ \cos \theta + \cos 180^\circ \sin \theta \overset{?}{=} -\sin \theta \quad \text{Sum of angles formula} \]

\[ 0 \cos \theta + (-1) \sin \theta \overset{?}{=} -\sin \theta \quad \text{Evaluate each expression.} \]

\[ -\sin \theta = -\sin \theta \quad \text{Simplify.} \]

\[ \text{b. } \cos (180^\circ + \theta) = -\cos \theta \]

\[ \cos (180^\circ + \theta) \overset{?}{=} -\cos \theta \quad \text{Original equation} \]

\[ \cos 180^\circ \cos \theta - \sin 180^\circ \sin \theta \overset{?}{=} -\cos \theta \quad \text{Sum of angles formula} \]

\[ (-1) \cos \theta - 0 \sin \theta \overset{?}{=} -\cos \theta \quad \text{Evaluate each expression.} \]

\[ -\cos \theta = -\cos \theta \quad \text{Simplify.} \]

**CHECK Your Progress**

3. Determine the amount of light energy in West Hollywood, California, which is located at a latitude of 34.1° N.

Verify that each of the following is an identity.

3A. \( \sin (90^\circ - \theta) = \cos \theta \)

3B. \( \cos (90^\circ + \theta) = -\sin \theta \)
Lesson 14-5 Sum and Differences of Angles Formulas

Example 1 (pp. 849–850)

Find the exact value of each expression.

1. \( \sin 75^\circ \)  
2. \( \sin 165^\circ \)  
3. \( \cos 255^\circ \)  
4. \( \cos (-30^\circ) \)  
5. \( \sin (-240^\circ) \)  
6. \( \cos (-120^\circ) \)  

Example 2 (p. 850)

Example 3 (p. 850)

Example 7: GEOMETRY Determine the exact value of \( \tan \alpha \) in the figure.

Verify that each of the following is an identity.

8. \( \cos (270^\circ - \theta) = -\sin \theta \)
9. \( \sin (\theta + \frac{\pi}{2}) = \cos \theta \)
10. \( \sin (\theta + 30^\circ) + \cos (\theta + 60^\circ) = \cos \theta \)

PHYSICS For Exercises 23–26, use the following information.

On December 22, the maximum amount of light energy that falls on a square foot of ground at a certain location is given by \( E \sin (113.5^\circ + \phi) \), where \( \phi \) is the latitude of the location. Find the amount of light energy, in terms of \( E \), for each location.

23. Salem, OR (Latitude: 44.9° N)  
24. Chicago, IL (Latitude: 41.8° N)  
25. Charleston, SC (Latitude: 28.5° N)  
26. San Diego, CA (Latitude: 32.7° N)

Verify that each of the following is an identity.

27. \( \sin (270^\circ - \theta) = -\cos \theta \)
28. \( \cos (90^\circ + \theta) = -\sin \theta \)
29. \( \cos (90^\circ - \theta) = \sin \theta \)
30. \( \sin (90^\circ - \theta) = \cos \theta \)
31. \( \sin (\theta + \frac{3\pi}{2}) = -\cos \theta \)
32. \( \cos (\pi - \theta) = -\cos \theta \)
33. \( \cos (2\pi + \theta) = \cos \theta \)
34. \( \sin (\pi - \theta) = \sin \theta \)

COMMUNICATION For Exercises 35 and 36, use the following information.

A radio transmitter sends out two signals, one for voice communication and another for data. Suppose the equation of the voice wave is \( v = 10 \sin (2t - 30^\circ) \) and the equation of the data wave is \( d = 10 \cos (2t + 60^\circ) \).

35. Draw a graph of the waves when they are combined.
36. Refer to the application at the beginning of the lesson. What type of interference results? Explain.

Verify that each of the following is an identity.

37. \( \sin (60^\circ + \theta) + \sin (60^\circ - \theta) = \sqrt{3} \cos \theta \)
38. \( \sin (\theta + \frac{\pi}{3}) - \cos (\theta + \frac{\pi}{6}) = \sin \theta \)
39. \( \sin (\alpha + \beta) \sin (\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta \)
40. \( \cos (\alpha + \beta) = \frac{1 - \tan \alpha \tan \beta}{\sec \alpha \sec \beta} \)
Verify that each of the following is an identity. (Lesson 14-4)

47. \( \cot \theta + \sec \theta = \frac{\cos^2 \theta + \sin \theta}{\sin \theta \cos \theta} \)

48. \( \sin^2 \theta + \tan^2 \theta = (1 - \cos^2 \theta) + \frac{\sec^2 \theta}{\csc^2 \theta} \)

49. \( \sin \theta (\sin \theta + \csc \theta) = 2 - \cos^2 \theta \)

50. \( \frac{\sec \theta}{\tan \theta} = \csc \theta \)

51. \( \frac{\tan \theta \csc \theta}{\sec \theta} \)

52. \( 4\left(\sec^2 \theta - \frac{\sin^2 \theta}{\cos^2 \theta}\right) \)

53. \( (\cot \theta + \tan \theta)\sin \theta \)

54. \( \csc \theta \tan \theta + \sec \theta \)

55. AVIATION A pilot is flying from Chicago to Columbus, a distance of 300 miles. In order to avoid an area of thunderstorms, she alters her initial course by 15° and flies on this course for 75 miles. How far is she from Columbus? (Lesson 13-5)

56. Write \( 6y^2 - 34x^2 = 204 \) in standard form. (Lesson 10-6)

PREREQUISITE SKILL Solve each equation. (Lesson 5-5)

57. \( x^2 = \frac{20}{16} \)

58. \( x^2 = \frac{9}{25} \)

59. \( x^2 = \frac{5}{25} \)

60. \( x^2 = \frac{18}{32} \)
Stringed instruments such as a piano, guitar, or violin rely on waves to produce the tones we hear. When the strings are struck or plucked, they vibrate. If the motion of the strings were observed in slow motion, you could see that there are places on the string, called nodes, that do not move under the vibration. Halfway between each pair of consecutive nodes are antinodes that undergo the maximum vibration. The nodes and antinodes form harmonics. These harmonics can be represented using variations of the equations \( y = \sin \theta \) and \( y = \sin \frac{\theta}{2} \).

### Double-Angle Formulas

You can use the formula for \( \sin(\alpha + \beta) \) to find the sine of twice an angle \( \theta \), \( \sin 2\theta \), and the formula for \( \cos(\alpha + \beta) \) to find the cosine of twice an angle \( \theta \), \( \cos 2\theta \).

\[
\sin 2\theta = \sin(\theta + \theta) = \sin \theta \cos \theta + \cos \theta \sin \theta = 2 \sin \theta \cos \theta
\]

\[
\cos 2\theta = \cos(\theta + \theta) = \cos \theta \cos \theta - \sin \theta \sin \theta = \cos^2 \theta - \sin^2 \theta
\]

You can find alternate forms for \( \cos 2\theta \) by making substitutions into the expression \( \cos^2 \theta - \sin^2 \theta \).

\[
\cos^2 \theta - \sin^2 \theta = (1 - \sin^2 \theta) - \sin^2 \theta \quad \text{Substitute } 1 - \sin^2 \theta \text{ for } \cos^2 \theta.
\]

\[
= 1 - 2 \sin^2 \theta \quad \text{Simplify.}
\]

\[
\cos^2 \theta - \sin^2 \theta = \cos^2 \theta - (1 - \cos^2 \theta) \quad \text{Substitute } 1 - \cos^2 \theta \text{ for } \sin^2 \theta.
\]

\[
= 2 \cos^2 \theta - 1 \quad \text{Simplify.}
\]

These formulas are called the double-angle formulas.

### Key Concept

The following identities hold true for all values of \( \theta \).

\[
\sin 2\theta = 2 \sin \theta \cos \theta \quad \cos 2\theta = \cos^2 \theta - \sin^2 \theta
\]

\[
\cos 2\theta = 1 - 2 \sin^2 \theta \quad \cos 2\theta = 2 \sin^2 \theta - 1
\]
You can derive formulas for the sine and cosine of half a given angle using the double-angle formulas.

Find \( \sin \frac{\alpha}{2} \).

\[ 1 - 2 \sin^2 \theta = \cos 2\theta \quad \text{Double-angle formula} \]
\[ 1 - 2 \sin^2 \frac{\alpha}{2} = \cos \alpha \quad \text{Substitute } \frac{\alpha}{2} \text{ for } \theta \text{ and } \alpha \text{ for } 2\theta. \]
\[ \sin^2 \frac{\alpha}{2} = \frac{1 - \cos \alpha}{2} \quad \text{Solve for } \sin^2 \frac{\alpha}{2}. \]
\[ \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} \quad \text{Take the square root of each side.} \]
Find \( \cos \frac{\alpha}{2} \).

\[
2 \cos^2 \theta - 1 = \cos 2\theta \quad \text{Double-angle formula}
\]

\[
2 \cos^2 \frac{\alpha}{2} - 1 = \cos \alpha \quad \text{Substitute } \frac{\alpha}{2} \text{ for } \theta \text{ and } \alpha \text{ for } 2\theta.
\]

\[
\cos^2 \frac{\alpha}{2} = \frac{1 + \cos \alpha}{2} \quad \text{Solve for } \cos^2 \frac{\alpha}{2}.
\]

\[
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{Take the square root of each side.}
\]

These are called the \textbf{half-angle formulas}. The signs are determined by the function of \( \frac{\alpha}{2} \).

**KEY CONCEPT**

<table>
<thead>
<tr>
<th>Half-Angle Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sin \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} )</td>
</tr>
<tr>
<td>( \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 - \cos \alpha}{2}} )</td>
</tr>
</tbody>
</table>

**EXAMPLE**

\textbf{Half-Angle Formulas}

Find \( \cos \frac{\alpha}{2} \) if \( \sin \alpha = -\frac{3}{4} \) and \( \alpha \) is in the third quadrant.

Since \( \cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \), we must find \( \cos \alpha \) first.

\[
\cos^2 \alpha = 1 - \sin^2 \alpha \quad \text{cos}^2 \alpha + \sin^2 \alpha = 1
\]

\[
\cos^2 \alpha = 1 - \left(-\frac{3}{4}\right)^2 \quad \sin \alpha = -\frac{3}{4}
\]

\[
\cos^2 \alpha = \frac{7}{16} \quad \text{Simplify.}
\]

\[
\cos \alpha = \pm \frac{\sqrt{7}}{4} \quad \text{Take the square root of each side.}
\]

Since \( \alpha \) is in the third quadrant, \( \cos \alpha = -\frac{\sqrt{7}}{4} \).

\[
\cos \frac{\alpha}{2} = \pm \sqrt{\frac{1 + \cos \alpha}{2}} \quad \text{Half-angle formula}
\]

\[
= \pm \sqrt{\frac{1 - \sqrt{7}}{4}} \quad \cos \alpha = -\frac{\sqrt{7}}{4}
\]

\[
= \pm \sqrt{\frac{4 - \sqrt{7}}{8}} \quad \text{Simplify the radicand.}
\]

\[
= \pm \frac{\sqrt{4 - \sqrt{7}} \cdot \sqrt{2}}{2\sqrt{2}} \quad \text{Rationalize.}
\]

\[
= \pm \frac{\sqrt{8 - 2\sqrt{7}}}{4} \quad \text{Multiply.}
\]

Since \( \alpha \) is between 180° and 270°, \( \frac{\alpha}{2} \) is between 90° and 135°. Thus, \( \cos \frac{\alpha}{2} \) is negative and equals \( -\frac{\sqrt{8 - 2\sqrt{7}}}{4} \).

2. Find \( \sin \frac{\alpha}{2} \) if \( \sin \alpha = \frac{2}{3} \) and \( \alpha \) is in the 2nd quadrant.
EXAMPLE Evaluate Using Half-Angle Formulas

Find the exact value of each expression by using the half-angle formulas.

a. \( \sin 105^\circ \)

\[
\sin 105^\circ = \sin \frac{210^\circ}{2} = \sqrt{\frac{1 - \cos 210^\circ}{2}} = \sqrt{\frac{1 - (-\sqrt{3}/2)}{2}} = \sqrt{\frac{2 + \sqrt{3}}{4}} = \frac{\sqrt{2 + \sqrt{3}}}{2}
\]

b. \( \cos \frac{\pi}{8} \)

\[
\cos \frac{\pi}{8} = \frac{\pi}{4} = \sqrt{\frac{1 + \cos \frac{\pi}{4}}{2}} = \sqrt{\frac{1 + \sqrt{2}/2}{2}} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}
\]

3A. \( \sin 135^\circ \)

3B. \( \cos \frac{7\pi}{8} \)

Recall that you can use the sum and difference formulas to verify identities. Double- and half-angle formulas can also be used to verify identities.

EXAMPLE Verify Identities

Verify that \((\sin \theta + \cos \theta)^2 = 1 + \sin 2\theta\) is an identity.

\[
(sin \theta + \cos \theta)^2 = 1 + \sin 2\theta \\
\sin^2 \theta + 2 \sin \theta \cos \theta + \cos^2 \theta = 1 + \sin 2\theta \\
1 + 2 \sin \theta \cos \theta = 1 + \sin 2\theta \\
1 + \sin 2\theta = 1 + \sin 2\theta
\]

4. Verify that \(4 \cos^2 x - \sin^2 2x = 4 \cos^4 x\)
Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

1. $\cos \theta = \frac{3}{5}; 0^\circ < \theta < 90^\circ$
2. $\cos \theta = -\frac{2}{3}; 180^\circ < \theta < 270^\circ$
3. $\sin \theta = \frac{1}{2}; 0^\circ < \theta < 90^\circ$
4. $\sin \theta = -\frac{3}{4}; 270^\circ < \theta < 360^\circ$

Find the exact value of each expression by using the half-angle formulas.

5. $\sin 195^\circ$
6. $\cos \frac{19\pi}{12}$

7. AVIATION When a jet travels at speeds greater than the speed of sound, a sonic boom is created by the sound waves forming a cone behind the jet. If $\theta$ is the measure of the angle at the vertex of the cone, then the Mach number $M$ can be determined using the formula $\sin \frac{\theta}{2} = \frac{1}{M}$. Find the Mach number of a jet if a sonic boom is created by a cone with a vertex angle of $75^\circ$.

Verify that each of the following is an identity.

8. $\cot x = \frac{\sin 2x}{1 - \cos 2x}$
9. $\cos^2 2x + 4 \sin^2 x \cos^2 x = 1$

Find the exact values of $\sin 2\theta$, $\cos 2\theta$, $\sin \frac{\theta}{2}$, and $\cos \frac{\theta}{2}$ for each of the following.

10. $\sin \theta = \frac{5}{13}; 90^\circ < \theta < 180^\circ$
11. $\cos \theta = \frac{1}{5}; 270^\circ < \theta < 360^\circ$
12. $\cos \theta = -\frac{1}{3}; 180^\circ < \theta < 270^\circ$
13. $\sin \theta = -\frac{3}{5}; 180^\circ < \theta < 270^\circ$
14. $\sin \theta = -\frac{3}{8}; 270^\circ < \theta < 360^\circ$
15. $\cos \theta = -\frac{1}{4}; 90^\circ < \theta < 180^\circ$

Find the exact value of each expression by using the half-angle formulas.

16. $\cos 165^\circ$
17. $\sin 22\frac{1}{2}^\circ$
18. $\cos 157\frac{1}{2}^\circ$
19. $\sin 345^\circ$
20. $\sin \frac{7\pi}{8}$
21. $\cos \frac{7\pi}{12}$

Verify that each of the following is an identity.

22. $\sin 2x = 2 \cot x \sin^2 x$
23. $2 \cos^2 \frac{x}{2} = 1 + \cos x$
24. $\sin^4 x - \cos^4 x = 2 \sin^2 x - 1$
25. $\sin^2 x = \frac{1}{2} (1 - \cos 2x)$
26. $\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}$
27. $\frac{1}{\sin x \cos x} - \frac{\cos x}{\sin x} = \tan x$
**PHYSICS** For Exercises 28 and 29, use the following information.

An object is propelled from ground level with an initial velocity of \( v \) at an angle of elevation \( \theta \).

28. The horizontal distance \( d \) it will travel can be determined using the formula \( d = \frac{v^2 \sin 2\theta}{g} \), where \( g \) is the acceleration due to gravity. Verify that this expression is the same as \( \frac{2}{g} v^2 (\tan \theta - \tan \sin^2 \theta) \).

29. The maximum height \( h \) the object will reach can be determined using the formula \( h = \frac{v^2 \sin^2 \theta}{2g} \). Find the ratio of the maximum height attained to the horizontal distance traveled.

Find the exact values of \( \sin 2\theta \), \( \cos 2\theta \), \( \sin \frac{\theta}{2} \), and \( \cos \frac{\theta}{2} \) for each of the following.

30. \( \cos \theta = \frac{1}{6}; 0^\circ < \theta < 90^\circ \)

31. \( \cos \theta = -\frac{12}{13}; 180^\circ < \theta < 270^\circ \)

32. \( \sin \theta = -\frac{1}{3}; 270^\circ < \theta < 360^\circ \)

33. \( \sin \theta = -\frac{1}{4}; 180^\circ < \theta < 270^\circ \)

34. \( \cos \theta = \frac{2}{3}; 0^\circ < \theta < 90^\circ \)

35. \( \sin \theta = \frac{2}{5}; 90^\circ < \theta < 180^\circ \)

36. **OPTICS** If a glass prism has an apex angle of measure \( \alpha \) and an angle of deviation of measure \( \beta \), then the index of refraction \( n \) of the prism is given by \( n = \frac{\sin \left( \frac{1}{2}(\alpha + \beta) \right)}{\sin \frac{\alpha}{2}} \). What is the angle of deviation of a prism with an apex angle of 40° and an index of refraction of 2?

**GEOGRAPHY** For Exercises 37 and 38, use the following information.

A Mercator projection map uses a flat projection of Earth in which the distance between the lines of latitude increases with their distance from the equator. The calculation of the location of a point on this projection uses the expression \( \tan \left( 45^\circ + \frac{L}{2} \right) \), where \( L \) is the latitude of the point.

37. Write this expression in terms of a trigonometric function of \( L \).

38. Find the exact value of the expression if \( L = 60^\circ \).

39. **REASONING** Explain how to find \( \cos \frac{x}{2} \) if \( x \) is in the third quadrant.

40. **REASONING** Describe the conditions under which you would use each of the three identities for \( \cos 2\theta \).

41. **OPEN ENDED** Find a counterexample to show that \( \cos 2\theta = 2 \cos \theta \) is not an identity.

42. **Writing in Math** Use the information on page 853 to explain how trigonometric functions can be used to describe music. Include a description of what happens to the graph of the function of a vibrating string as it moves from one harmonic to the next and an explanation of what happens to the period of the function as you move from the \( n \)th harmonic to the (\( n + 1 \))th harmonic.
Lesson 14-6  Double-Angle and Half-Angle Formulas

43. ACT/SAT  Find the exact value of \( \cos 2\theta \)
if \( \sin \theta = -\frac{\sqrt{5}}{3} \) and \( 180^\circ < \theta < 270^\circ \).
A  \( -\frac{\sqrt{6}}{6} \)
B  \( -\frac{\sqrt{30}}{6} \)
C  \( -\frac{4\sqrt{5}}{9} \)
D  \( -\frac{1}{9} \)

44. REVIEW  Which of the following is equivalent to \( \frac{\cos \theta (\cot^2 \theta + 1)}{\csc \theta} \)?
F  \( \tan \theta \)
G  \( \cot \theta \)
H  \( \sec \theta \)
J  \( \csc \theta \)

Find the exact value of each expression.  (Lesson 14-5)

45.  \( \cos 15^\circ \)
46.  \( \sin 15^\circ \)
47.  \( \sin (-135^\circ) \)
48.  \( \cos 150^\circ \)
49.  \( \sin 105^\circ \)
50.  \( \cos (-300^\circ) \)

Verify that each of the following is an identity.  (Lesson 14-4)

51.  \( \cot^2 \theta - \sin^2 \theta = \frac{\cos^2 \theta \csc^2 \theta - \sin^2 \theta}{\sin^2 \theta \csc^2 \theta} \)
52.  \( \cos \theta (\cos \theta + \cot \theta) = \cot \theta \cos \theta (\sin \theta + 1) \)

ANALYZE TABLES  For Exercises 53 and 54, use the following information.

The magnitude of an earthquake \( M \) measured on the Richter scale is given by \( M = \log_{10} x \), where \( x \) represents the amplitude of the seismic wave causing ground motion.  (Lesson 9-2)

53.  How many times as great was the 1960 Chile earthquake as the 1938 Indonesia earthquake?
54.  The largest aftershock of the 1964 Alaskan earthquake was 6.7 on the Richter scale. How many times as great was the main earthquake as this aftershock?

Write each expression in quadratic form, if possible.  (Lesson 6-6)

55.  \( a^8 - 7a^4 + 13 \)
56.  \( 5n^7 + 3n - 3 \)
57.  \( d^6 + 2d^3 + 10 \)

Find each value if \( f(x) = x^2 - 7x + 5 \).  (Lesson 2-1)

58.  \( f(2) \)
59.  \( f(0) \)
60.  \( f(-3) \)
61.  \( f(n) \)

PREREQUISITE SKILL  Solve each equation.  (Lesson 5-3)

62.  \( (x + 6)(x - 5) = 0 \)
63.  \( (x - 1)(x + 1) = 0 \)
64.  \( x(x + 2) = 0 \)
65.  \( (2x - 5)(x + 2) = 0 \)
66.  \( (2x + 1)(2x - 1) = 0 \)
67.  \( x^2(2x + 1) = 0 \)
The graph of a trigonometric function is made up of points that represent all values that satisfy the function. To solve a trigonometric equation, you need to find all values of the variable that satisfy the equation. You can use a TI-83/84 Plus to solve trigonometric equations by graphing each side of the equation as a function and then locating the points of intersection.

**ACTIVITY 1**

Use a graphing calculator to solve \( \sin x = 0.2 \) if \( 0^\circ \leq x < 360^\circ \).

Rewrite the equation as two functions, \( y = \sin x \) and \( y = 0.2 \). Then graph the two functions. Look for the point of intersection.

**KEYSTROKES:**

```
MODE \n360 \n90 \n-2 \n1 \nENTER \nY= \nSIN \nX,T,0,n \nENTER \n0.2 \nENTER \nGRAPH
```

![Graph showing two functions intersecting at two points](Image)

Based on the graph, you can see that there are two points of intersection in the interval \( 0^\circ \leq x < 360^\circ \). Use \([\text{ZOOM}] \) or \([\text{CALC}] \) 5 to approximate the solutions. The approximate solutions are 168.5° and 11.5°.

Like other equations you have studied, some trigonometric equations have no real solutions. Carefully examine the graphs over their respective periods for points of intersection. If there are no points of intersection, then the trigonometric equation has no real solutions.

**ACTIVITY 2**

Use a graphing calculator to solve \( \tan^2 x \cos x + 5 \cos x = 0 \) if \( 0^\circ \leq x < 360^\circ \).

Because the tangent function is not continuous, place the calculator in Dot mode. The related functions to be graphed are \( y = \tan^2 x \cos x + 5 \cos x \) and \( y = 0 \).

These two functions do not intersect. Therefore, the equation \( \tan^2 x \cos x + 5 \cos x = 0 \) has no real solutions.

**EXERCISES**

Use a graphing calculator to solve each equation for the values of \( x \) indicated.

1. \( \sin x = 0.8 \) if \( 0^\circ \leq x < 360^\circ \)
2. \( \tan x = \sin x \) if \( 0^\circ \leq x < 360^\circ \)
3. \( 2 \cos x + 3 = 0 \) if \( 0^\circ \leq x < 360^\circ \)
4. \( 0.5 \cos x = 1.4 \) if \( -720^\circ \leq x < 720^\circ \)
5. \( \sin 2x = \sin x \) if \( 0^\circ \leq x < 360^\circ \)
6. \( \sin 2x - 3 \sin x = 0 \) if \( -360^\circ \leq x < 360^\circ \)
Solve Trigonometric Equations

The average daily high temperature for a region can be described by a trigonometric function. For example, the average daily high temperature for each month in Orlando, Florida, can be modeled by the function

\[ T = 11.56 \sin(0.4516x - 1.641) + 80.89, \]

where \( T \) represents the average daily high temperature in degrees Fahrenheit and \( x \) represents the month of the year. This equation can be used to predict the months in which the average temperature in Orlando will be at or above a desired temperature.

**Solve Trigonometric Equations**

You have seen that trigonometric identities are true for all values of the variable for which the equation is defined. However, most trigonometric equations, like some algebraic equations, are true for some but not all values of the variable.

**EXAMPLE**

**Solve Equations for a Given Interval**

Find all solutions of \( \sin 2\theta = 2 \cos \theta \) for the interval \( 0 \leq \theta < 360^\circ \).

\[
\begin{align*}
\sin 2\theta &= 2 \cos \theta & \text{Original equation} \\
2 \sin \theta \cos \theta &= 2 \cos \theta & \sin 2\theta = 2 \sin \theta \cos \theta \\
2 \sin \theta \cos \theta - 2 \cos \theta &= 0 & \text{Solve for 0.} \\
2 \cos \theta (\sin \theta - 1) &= 0 & \text{Factor.} \\
\end{align*}
\]

Use the Zero Product Property.

\[
\begin{align*}
2 \cos \theta &= 0 & \text{or} & \sin \theta - 1 &= 0 \\
\cos \theta &= 0 & \sin \theta &= 1 \\
\theta &= 90^\circ \text{ or } 270^\circ & \theta &= 90^\circ \\
\end{align*}
\]

The solutions are 90° and 270°.

1. Find all solutions of \( \cos^2 \theta = 1 \) for the interval \( 0^\circ \leq \theta < 360^\circ \).
Trigonometric equations are usually solved for values of the variable between $0^\circ$ and $360^\circ$ or $0$ radians and $2\pi$ radians. There are solutions outside that interval. These other solutions differ by integral multiples of the period of the function.

**EXAMPLE** Solve Trigonometric Equations

Solve $2 \sin \theta = -1$ for all values of $\theta$ if $\theta$ is measured in radians.

1. $2 \sin \theta = -1$  \hspace{1cm} \text{Original equation}
2. $\sin \theta = -\frac{1}{2}$  \hspace{1cm} \text{Divide each side by 2.}

Look at the graph of $y = \sin \theta$ to find solutions of $\sin \theta = -\frac{1}{2}$.

The solutions are $\frac{7\pi}{6}$, $\frac{11\pi}{6}$, $\frac{19\pi}{6}$, $\frac{23\pi}{6}$, and so on, and $\frac{-7\pi}{6}$, $\frac{-11\pi}{6}$, $\frac{-19\pi}{6}$, $\frac{-23\pi}{6}$, and so on. The only solutions in the interval $0$ to $2\pi$ are $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. The period of the sine function is $2\pi$ radians. So the solutions can be written as $\frac{7\pi}{6} + 2k\pi$ and $\frac{11\pi}{6} + 2k\pi$, where $k$ is any integer.

**CHECK YOUR PROGRESS**

2. Solve for $\cos 2\theta + \cos \theta + 1 = 0$ for all values of $\theta$ if $\theta$ is measured in degrees.

If an equation cannot be solved easily by factoring, try rewriting the expression using trigonometric identities. However, using identities and some algebraic operations, such as squaring, may result in extraneous solutions. So, it is necessary to check your solutions using the original equation.

**EXAMPLE** Solve Trigonometric Equations Using Identities

Solve $\cos \theta \tan \theta - \sin^2 \theta = 0$.

1. $\cos \theta \left(\frac{\sin \theta}{\cos \theta}\right) - \sin^2 \theta = 0$  \hspace{1cm} \text{Multiply.}
2. $\sin \theta - \sin^2 \theta = 0$  \hspace{1cm} \text{Factor.}
3. $\sin \theta (1 - \sin \theta) = 0$  \hspace{1cm} \text{by zero product property.}
4. $\sin \theta = 0$  \hspace{1cm} \text{or}  \hspace{1cm} 1 - \sin \theta = 0$
5. $\theta = 0^\circ$, $180^\circ$, or $360^\circ$  \hspace{1cm} $\sin \theta = 1$
6. $\theta = 90^\circ$  \hspace{1cm} $1 - \sin \theta = 0$
Determine Whether a Solution Exists

Solve $3 \cos 2 \theta - 5 \cos \theta = 1$.

Original equation:

$3 \cos 2 \theta - 5 \cos \theta = 1$

Multiply:

$6 \cos^2 \theta - 3 - 5 \cos \theta = 1$

Subtract 1 from each side:

$6 \cos^2 \theta - 5 \cos \theta - 4 = 0$

Factor:

$(3 \cos \theta - 4)(2 \cos \theta + 1) = 0$

$3 \cos \theta - 4 = 0$ or $2 \cos \theta + 1 = 0$

$3 \cos \theta = 4$ or $2 \cos \theta = -1$

$\cos \theta = \frac{4}{3}$ or $\cos \theta = -\frac{1}{2}$

Not possible since $\cos \theta$ cannot be greater than 1.

Thus, the solutions are $120^\circ + k \cdot 360^\circ$ and $240^\circ + k \cdot 360^\circ$.

CHECK Your Progress

Solve each equation.

3A. $\sin \theta \cot \theta - \cos^2 \theta = 0$

3B. $\frac{\cos \theta}{\cot \theta} + 2 \sin^2 \theta = 0$

Some trigonometric equations have no solution. For example, the equation $\cos x = 4$ has no solution since all values of $\cos x$ are between $-1$ and $1$, inclusive.

Thus, the solution set for $\cos x = 4$ is empty.

4 Solve $3 \cos 2 \theta - 5 \cos \theta = 1$.

$3 \cos 2 \theta - 5 \cos \theta = 1$

$3(2 \cos^2 \theta - 1) - 5 \cos \theta = 1$

$6 \cos^2 \theta - 3 - 5 \cos \theta = 1$

$6 \cos^2 \theta - 5 \cos \theta - 4 = 0$

$(3 \cos \theta - 4)(2 \cos \theta + 1) = 0$

$3 \cos \theta - 4 = 0$ or $2 \cos \theta + 1 = 0$

$3 \cos \theta = 4$ or $2 \cos \theta = -1$

$\cos \theta = \frac{4}{3}$ or $\cos \theta = -\frac{1}{2}$

Not possible since $\cos \theta$ cannot be greater than 1.

Thus, the solutions are $120^\circ + k \cdot 360^\circ$ and $240^\circ + k \cdot 360^\circ$.

CHECK Your Progress

Solve each equation.

4A. $\sin^2 \theta + 2 \cos^2 \theta = 4$

4B. $\cos^2 \theta - 3 = 4 - \sin^2 \theta$
Use Trigonometric Equations  Trigonometric equations are often used to solve real-world situations.

**Real-World EXAMPLE**

**GARDENING** Rhonda wants to wait to plant her flowers until there are at least 14 hours of daylight. The number of hours of daylight $H$ in her town can be represented by $H = 11.45 + 6.5 \sin(0.0168d - 1.333)$, where $d$ is the day of the year and angle measures are in radians. On what day is it safe for Rhonda to plant her flowers?

1. $H = 11.45 + 6.5 \sin(0.0168d - 1.333)$  
   \text{Original equation}

2. $14 = 11.45 + 6.5 \sin(0.0168d - 1.333)$  
   \text{$H = 14$}

3. $2.55 = 6.5 \sin(0.0168d - 1.333)$  
   \text{Subtract 11.45 from each side.}

4. $0.392 = \sin(0.0168d - 1.333)$  
   \text{Divide each side by 6.5.}

5. $0.403 = 0.0168d - 1.333$  
   \text{sin}^{-1} 0.392 = 0.403

6. $1.736 = 0.0168d$  
   \text{Add 1.333 to each side.}

7. $103.333 = d$  
   \text{Divide each side by 0.0168.}

Rhonda can safely plant her flowers around the 104th day of the year, or around April 14.

**CHECK Your Progress**

5. If Rhonda decides to wait only until there are 12 hours of daylight, on what day is it safe for her to plant her flowers?

---

Find all solutions of each equation for the given interval.

1. $4 \cos^2 \theta = 1; 0^\circ \leq \theta < 360^\circ$
2. $2 \sin^2 \theta - 1 = 0; 90^\circ < \theta < 270^\circ$
3. $\sin 2\theta = \cos \theta; 0 \leq \theta < 2\pi$
4. $3 \sin^2 \theta - \cos^2 \theta = 0; 0 \leq \theta < \frac{\pi}{2}$

Solve each equation for all values of $\theta$ if $\theta$ is measured in radians.

5. $\cos 2\theta = \cos \theta$
6. $\sin \theta + \sin \theta \cos \theta = 0$

Solve each equation for all values of $\theta$ if $\theta$ is measured in degrees.

7. $\sin \theta = 1 + \cos \theta$
8. $2 \cos^2 \theta + 2 = 5 \cos \theta$

Solve each equation for all values of $\theta$.

9. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$
10. $2 \cos^2 \theta + 3 \sin \theta - 3 = 0$

11. **PHYSICS** According to Snell’s law, the angle at which light enters water $\alpha$ is related to the angle at which light travels in water $\beta$ by the equation $\sin \alpha = 1.33 \sin \beta$. At what angle does a beam of light enter the water if the beam travels at an angle of $23^\circ$ through the water?
Find all solutions of each equation for the given interval.

12. $2 \cos \theta - 1 = 0; 0^\circ \leq \theta < 360^\circ$
13. $2 \sin \theta = -\sqrt{3}; 180^\circ < \theta < 360^\circ$
14. $4 \sin^2 \theta = 1; 180^\circ < \theta < 360^\circ$
15. $4 \cos^2 \theta = 3; 0^\circ \leq \theta < 360^\circ$

Solve each equation for all values of $\theta$ if $\theta$ is measured in radians.

16. $\cos 2\theta + 3 \cos \theta - 1 = 0$
17. $2 \sin^2 \theta - \cos \theta - 1 = 0$
18. $\cos^2 \theta - \frac{5}{2} \cos \theta - \frac{3}{2} = 0$
19. $\cos \theta = 3 \cos \theta - 2$

Solve each equation for all values of $\theta$ if $\theta$ is measured in degrees.

20. $\sin \theta = \cos \theta$
21. $\tan \theta = \sin \theta$
22. $\sin^2 \theta - 2 \sin \theta - 3 = 0$
23. $4 \sin^2 \theta - 4 \sin \theta + 1 = 0$

Solve each equation for all values of $\theta$.

24. $\sin^2 \theta + \cos 2\theta - \cos \theta = 0$
25. $2 \sin^2 \theta - 3 \sin \theta - 2 = 0$
26. $\sin^2 \theta = \cos^2 \theta - 1$
27. $2 \cos^2 \theta + \cos \theta = 0$

**WAVES** For Exercises 28 and 29, use the following information.

After a wave is created by a boat, the height of the wave can be modeled using $y = \frac{1}{2}h + \frac{1}{2}h \sin \frac{2\pi t}{P}$, where $h$ is the maximum height of the wave in feet, $P$ is the period in seconds, and $t$ is the propagation of the wave in seconds.

28. If $h = 3$ and $P = 2$, write the equation for the wave and draw its graph over a 10-second interval.

29. How many times over the first 10 seconds does the graph predict the wave to be one foot high?

Find all solutions of each equation for the given interval.

30. $2 \cos^2 \theta = \sin \theta + 1; 0 \leq \theta < 2\pi$
31. $\sin^2 \theta - 1 = \cos^2 \theta; 0 \leq \theta < \pi$
32. $2 \sin^2 \theta + \sin \theta = 0; \pi < \theta < 2\pi$
33. $2 \cos^2 \theta = -\cos \theta; 0 \leq \theta < 2\pi$

Solve each equation for all values of $\theta$ if $\theta$ is measured in radians.

34. $4 \cos^2 \theta - 4 \cos \theta + 1 = 0$
35. $\cos 2\theta = 1 - \sin \theta$
36. $(\cos \theta)(\sin 2\theta) - 2 \sin \theta + 2 = 0$
37. $2 \sin^2 \theta + (\sqrt{2} - 1) \sin \theta = \frac{\sqrt{2}}{2}$

Solve each equation for all values of $\theta$ if $\theta$ is measured in degrees.

38. $\tan^2 \theta - \sqrt{3} \tan \theta = 0$
39. $\cos^2 \theta - \frac{7}{2} \cos \theta - 2 = 0$
40. $\sin 2\theta + \frac{\sqrt{3}}{2} = \sqrt{3} \sin \theta + \cos \theta$
41. $1 - \sin^2 \theta - \cos \theta = \frac{3}{4}$

Solve each equation for all values of $\theta$.

42. $\sin \frac{\theta}{2} + \cos \theta = 1$
43. $\sin \frac{\theta}{2} + \cos \frac{\theta}{2} = \sqrt{2}$
44. $2 \sin \theta = \sin 2\theta$
45. $\tan^2 \theta + \sqrt{3} = (1 + \sqrt{3}) \tan \theta$

**LIGHT** For Exercises 46 and 47, use the following information.

The height of the International Peace Memorial at Put-in-Bay, Ohio, is 352 feet.

46. The length of the shadow $S$ of the Memorial depends upon the angle of inclination of the Sun, $\theta$. Express $S$ as a function of $\theta$.

47. Find the angle of inclination $\theta$ that will produce a shadow 560 feet long.
Find the exact value of \( \sin 2\theta \), \( \cos 2\theta \), \( \sin \frac{\theta}{2} \), and \( \cos \frac{\theta}{2} \) for each of the following.

(Lesson 14-6)

55. \( \sin \theta = \frac{3}{5} \); \( 0^\circ < \theta < 90^\circ \)

57. \( \cos \theta = \frac{5}{6} \); \( 0^\circ < \theta < 90^\circ \)

Find the exact value of each expression. (Lesson 14-5)

59. \( \sin 240^\circ \) 

60. \( \cos 315^\circ \) 

61. \( \sin 150^\circ \) 

62. Solve \( \triangle ABC \). Round measures of sides and angles to the nearest tenth. (Lesson 13-4)

**Cross-Curricular Project**

**Algebra and Physics**

*So you want to be a rocket scientist?* It is time to complete your project. Use the information and data you have gathered about the applications of trigonometry to prepare a poster, report, or Web page. Be sure to include graphs, tables, or diagrams in the presentation.

MathOnline Cross-Curricular Project at algebra2.com
Key Vocabulary

amplitude (p. 823) sum of angles formula (p. 849)
difference of angles formula (p. 849) trigonometric equation (p. 861)
double-angle formula (p. 853) trigonometric identity (p. 837)
half-angle formula (p. 855) vertical shift (p. 831)
midline (p. 831) phase shift (p. 829)

Vocabulary Check

Choose the correct term from the list above to complete each sentence.

1. The horizontal translation of a trigonometric function is a(n) __________.
2. A reference line about which a graph oscillates is a(n) __________.
3. The vertical translation of a trigonometric function is called a(n) __________.
4. The __________ formula can be used to find cos 22°.
5. The __________ can be used to find sin 60° using 30° as a reference.
6. The __________ can be used to find the sine or cosine of 75° if the sine and cosine of 45° and 30° are known.
7. A(n) __________ is an equation that is true for all values for which every expression in the equation is defined.
8. The __________ can be used to find the sine or cosine of 65° if the sine and cosine of 90° and 25° are known.
9. The absolute value of half the difference between the maximum value and the minimum value of a periodic function is called the __________.
Lesson-by-Lesson Review

14–1 Graphing Trigonometric Functions (pp. 822–828)

Find the amplitude, if it exists, and period of each function. Then graph each function.

10. \( y = -\frac{1}{2} \cos \theta \)
11. \( y = 4 \sin 2\theta \)
12. \( y = \sin \frac{1}{2} \theta \)
13. \( y = 5 \sec \theta \)
14. \( y = \frac{1}{2} \csc \frac{2}{3} \theta \)
15. \( y = \tan 4\theta \)

16. MECHANICS The position of a piston can be modeled using the equation \( y = A \sin \left( \frac{1}{4} \cdot 2\pi t \right) \) where \( A \) is the amplitude of oscillation and \( t \) is the time in seconds. Determine the period of oscillation.

Example 1 Find the amplitude and period of \( y = 2 \cos 4\theta \). Then graph.

- The amplitude is \( |2| \) or 2.
- The period is \( \frac{360^\circ}{4} \) or 90°.

Use the amplitude and period to graph the function.

14–2 Translations of Trigonometric Graphs (pp. 829–836)

State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

17. \( y = \frac{1}{2} \sin \left[ 2(\theta - 60^\circ) \right] - 1 \)
18. \( y = 2 \tan \left[ \frac{1}{4}(\theta - 90^\circ) \right] + 3 \)
19. \( y = 3 \sec \left[ \frac{1}{2}(\theta + \frac{\pi}{4}) \right] + 1 \)
20. \( y = \frac{1}{3} \cos \left[ \frac{1}{3}(\theta - \frac{2\pi}{3}) \right] - 2 \)

21. BIOLOGY The population of a species of bees varies periodically over the course of a year. The maximum population of bees occurs in March, and is 50,000. The minimum population of bees occurs in September and is 20,000. Assume that the population can be modeled using the sine function. Write an equation to represent the population of bees \( p \), \( t \) months after January.

Example 2 State the vertical shift, amplitude, period, and phase shift of \( y = 3 \sin \left[ 2 \left( \theta - \frac{\pi}{2} \right) \right] - 2 \). Then graph.

- Identify the values of \( k, a, b, \) and \( h \).
- \( k = -2 \), so the vertical shift is \(-2\).
- \( a = 3 \), so the amplitude is 3.
- \( b = 2 \), so the period is \( \frac{2\pi}{2} \) or \( \pi \).
- \( h = \frac{\pi}{2} \), so the phase shift is \( \frac{\pi}{2} \) to the right.
14–3
Trigonometric Identities (pp. 837–841)

Find the value of each expression.

22. \( \cot \theta \), if \( \csc \theta = -\frac{5}{3}; \) \( 270^\circ < \theta < 360^\circ \)
23. \( \sec \theta \), if \( \sin \theta = \frac{1}{2}; \) \( 0^\circ \leq \theta < 90^\circ \)

Simplify each expression.

24. \( \sin \theta \csc \theta - \cos^2 \theta \)
25. \( \cos^2 \theta \sec \theta \csc \theta \)
26. \( \cos \theta + \sin \theta \tan \theta \)
27. \( \sin \theta (1 + \cot^2 \theta) \)

28. PHYSICS The magnetic force on a particle can be modeled by the equation \( F = qvB \sin \theta \), where \( F \) is the magnetic force, \( q \) is the charge of the particle, \( B \) is the magnetic field strength, and \( \theta \) is the angle between the particle’s path and the direction of the magnetic field. Write an equation for the magnetic force in terms of \( \tan \theta \) and \( \sec \theta \).

Example 3 Find \( \cos \theta \) if \( \sin \theta = -\frac{3}{4} \) and \( 90^\circ < \theta < 180^\circ \).

\[
\begin{align*}
\cos^2 \theta + \sin^2 \theta &= 1 \\
\cos^2 \theta &= 1 - \sin^2 \theta \\
\cos^2 \theta &= 1 - \left(\frac{3}{4}\right)^2 \\
\cos^2 \theta &= 1 - \frac{9}{16} \\
\cos^2 \theta &= \frac{7}{16} \\
\cos \theta &= \pm \frac{\sqrt{7}}{4}
\end{align*}
\]

Since \( \theta \) is in the second quadrant, \( \cos \theta \) is negative. Thus, \( \cos \theta = -\frac{\sqrt{7}}{4} \).

Example 4 Simplify \( \sin \theta \cot \theta \cos \theta \).

\[
\begin{align*}
\sin \theta \cot \theta \cos \theta &= \frac{\sin \theta \cdot \cos \theta \cdot \cos \theta}{1} \\
&= \cos^2 \theta
\end{align*}
\]

Multiply.

Example 5 Verify that \( \tan \theta + \cot \theta = \sec \theta \csc \theta \) is an identity.

\[
\begin{align*}
\tan \theta + \cot \theta &= \frac{\sin \theta}{\cos \theta} + \frac{\cos \theta}{\sin \theta} \\
&= \frac{\sin \theta \cdot \sin \theta + \cos \theta \cdot \cos \theta}{\cos \theta \sin \theta} \\
&= \frac{\cos^2 \theta + \sin^2 \theta}{\cos \theta \sin \theta} \\
&= \frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}
\end{align*}
\]

Rewrite using the LCD, \( \cos \theta \sin \theta \).

\[
\begin{align*}
\frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta} &= \frac{1}{\cos \theta \sin \theta} \\
&= \frac{1}{\cos \theta} \cdot \frac{1}{\sin \theta}
\end{align*}
\]

Rewrite as the product of two expressions.

\[
\begin{align*}
\sec \theta \csc \theta &= \sec \theta \csc \theta \\
\frac{1}{\cos \theta} &= \sec \theta, \\
\frac{1}{\sin \theta} &= \csc \theta
\end{align*}
\]

14–4
Verifying Trigonometric Identities (pp. 842–846)

Verify that each of the following is an identity.

29. \( \frac{\sin \theta}{\tan \theta} + \frac{\cos \theta}{\cot \theta} = \cos \theta + \sin \theta \)
30. \( \frac{\sin \theta}{1 - \cos \theta} = \csc \theta + \cot \theta \)
31. \( \cot^2 \theta \sec^2 \theta = 1 + \cot^2 \theta \)
32. \( \sec \theta (\sec \theta - \cos \theta) = \tan^2 \theta \)

33. OPTICS The amount of light passing through a polarization filter can be modeled using the equation \( I = I_m \cos^2 \theta \), where \( I \) is the amount of light passing through the filter, \( I_m \) is the amount of light shined on the filter, and \( \theta \) is the angle of rotation between the light source and the filter. Verify the identity \( I_m \cos^2 \theta = I_m - \frac{I_m}{\cos^2 \theta + 1} \).
Find all solutions of each equation for the interval $0^\circ \leq \theta < 360^\circ$.

48. $2 \sin 2\theta = 1$

49. $\cos^2 \theta + \sin^2 \theta = 2 \cos \theta$

50. **PRISMS** The horizontal and vertical components of an oblique prism can be modeled using the equations $Z_x = P \cos \theta$ and $Z_y = P \sin \theta$ where $Z_x$ is the horizontal component, $Z_y$ is the vertical component, $P$ is the power of the prism, and $\theta$ is the angle between the prism and the horizontal. For what values of $\theta$ will the vertical and horizontal components be equivalent?
State the vertical shift, amplitude, period, and phase shift of each function. Then graph the function.

1. \( y = \frac{2}{3} \sin 2\theta + 5 \)
2. \( y = 4 \cos \left[ \frac{1}{2}(\theta + 30^\circ) \right] - 1 \)
3. \( y = 7 \cos \left[ 4\left(\theta + \frac{\pi}{6}\right) \right] \)

4. **AUTOMOTIVE** The pistons in a car oscillate according to a sine function. The amplitude of the oscillation is 2, the period is \( 6\pi \), and the phase shift is \( \frac{\pi}{2} \) to the left. Write a formula to model the position of the piston, \( p \), at time \( t \) seconds. Graph the equation.

Find the exact value of each expression.

13. \( \cos 165^\circ \)
14. \( \sin 255^\circ \)
15. \( \sin(-225^\circ) \)
16. \( \cos 480^\circ \)
17. \( \cos 67.5^\circ \)
18. \( \sin 75^\circ \)

Solve each equation for all values of \( \theta \) if \( \theta \) is measured in degrees.

19. \( \sec \theta = 1 + \tan \theta \)
20. \( \cos 2\theta = \cos \theta \)
21. \( \cos 2\theta + \sin \theta = 1 \)
22. \( \sin \theta = \tan \theta \)

**GOLF** For Exercises 23 and 24, use the following information.

A golf ball leaves the club with an initial velocity of 100 feet per second. The distance the ball travels is found by the formula

\[ d = \frac{v_0^2}{g} \sin 2\theta \]

where \( v_0 \) is the initial velocity, \( g \) is the acceleration due to gravity, and \( \theta \) is the measurement of the angle that the path of the ball makes with the ground. The acceleration due to gravity is 32 feet per second squared.

23. Find the distance that the ball travels if the angle between the path of the ball and the ground measures 60°.
24. If a ball travels 312.5 feet, what was the angle the path of the ball made with the ground to the nearest degree?

25. **MULTIPLE CHOICE** Identify the equation of the graphed function.

\[ A \quad y = 3 \cos 2\theta \quad C \quad y = 3 \cos \frac{1}{2}\theta \]
\[ B \quad y = \frac{1}{3} \cos 2\theta \quad D \quad y = \frac{1}{3} \cos \frac{1}{2}\theta \]
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. A small business owner must hire seasonal workers as the need arises. The following list shows the number of employees hired monthly for a 5-month period.
   
   5, 14, 6, 8, 12

   If the mean of this data is 9, what is the population standard deviation for these data? (Round to the nearest tenth.)
   
   A 2.6  
   B 5.7  
   C 8.6  
   D 12.3  

2. If \( f(x) = 2x^3 + 5x - 8 \), find \( f(2a^2) \).
   
   F \( f(2a^2) = 16a^5 + 10a^2 - 8 \)  
   G \( f(2a^2) = 64a^5 + 10a^2 - 8 \)  
   H \( f(2a^2) = 16a^6 + 10a^2 - 8 \)  
   J \( f(2a^2) = 64a^6 + 10a^2 - 8 \)  

3. Simplify \( 128^{\frac{1}{2}} \).
   
   A \( 2\sqrt{2} \)  
   B \( 2\sqrt{8} \)  
   C 4  
   D 4\( \sqrt{2} \)  

4. Lisa is 6 years younger than Petra. Stella is twice as old as Petra. The total of their ages is 54. Which equation can be used to find Petra’s age?
   
   F \( x + (x - 6) + 2(x - 6) = 54 \)  
   G \( x - 6x + (x + 2) = 54 \)  
   H \( x - 6 + 2x = 54 \)  
   J \( x + (x - 6) + 2x = 54 \)  

5. GRIDDABLE The mean of seven numbers is 0. The sum of three of the numbers is -9. What is the sum of the remaining four numbers?

6. Which of the following functions represents exponential decay?
   
   A \( y = 0.2(7)^x \)  
   B \( y = (0.5)^x \)  
   C \( y = 4(9)^x \)  
   D \( y = 5\left(\frac{4}{3}\right)^x \)  

7. Solve the following system of equations.
   
   \[
   \begin{align*}
   3y &= 4x + 1 \\
   2y - 3x &= 2 
   \end{align*}
   \]
   
   F \((-4, -5)\)  
   G \((-2, -3)\)  
   H \((2, 3)\)  
   J \((4, 5)\)  

8. GRIDDABLE If \( k \) is a positive integer and \( 7k + 3 \) equals a prime number that is less than 50, then what is one possible value of \( 7k + 3 \)?
9. Find the center and radius of the circle with the equation \((x - 4)^2 + y^2 - 16 = 0\).

A \(C(-4, 0); r = 4\) units
B \(C(-4, 0); r = 16\) units
C \(C(4, 0); r = 4\) units
D \(C(4, 0); r = 16\) units

10. There are 16 green marbles, 2 red marbles, and 6 yellow marbles in a jar. How many yellow marbles need to be added to the jar in order to double the probability of selecting a yellow marble?

F 4
G 6
H 8
J 12

11. What is the effect of the graph on the equation \(y = 3x^2\) when the equation is changed to \(y = 2x^2\)?

A The graph of \(y = 2x^2\) is a reflection of the graph of \(y = 3x^2\) across the \(y\)-axis.
B The graph is rotated 90 degrees about the origin.
C The graph is narrower.
D The graph is wider.

12. GEOMETRY The perimeter of a right triangle is 36 inches. Twice the length of the longer leg minus twice the length of the shorter leg exceeds the hypotenuse by 6 inches. What are the lengths of all three sides?

F 3 in., 4 in., 5 in.
G 6 in., 8 in., 10 in.
H 9 in., 12 in., 15 in.
J 12 in., 16 in., 20 in.

13. The table below shows the cost of a pizza depending on the diameter of the pizza.

<table>
<thead>
<tr>
<th>Dimensions</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Round 10” diameter</td>
<td>8.10</td>
</tr>
<tr>
<td>Round 20” diameter</td>
<td>15.00</td>
</tr>
<tr>
<td>Square 10” side</td>
<td>10.00</td>
</tr>
<tr>
<td>Square 20” side</td>
<td>20.00</td>
</tr>
</tbody>
</table>

Which pizza should you buy if you want to get the most pizza per dollar?