Focus
Use multiple representations, technology, applications and modeling, and numerical fluency in discrete problem-solving contexts.

CHAPTER 11
Sequences and Series
BIG Idea Use sequences and series as well as tools and technology to represent, analyze, and solve real-life problems.

CHAPTER 12
Probability and Statistics
BIG Idea Use probability and statistical models to describe everyday situations involving chance.
Cross-Curricular Project

Algebra and Social Studies

**Math from the Past**  Emmy Noether was a German-born mathematician and professor who taught in Germany and the United States. She made important contributions in both mathematics and physics. In this project, you will research a mathematician of the past and his or her role in the development of discrete mathematics.

**Math Online**  Log on to algebra2.com to begin.
Sequences and Series

BIG Ideas
- Use arithmetic and geometric sequences and series.
- Use special sequences and iterate functions.
- Expand powers by using the Binomial Theorem.
- Prove statements by using mathematical induction.

Key Vocabulary
- arithmetic sequence (p. 622)
- arithmetic series (p. 629)
- geometric sequence (p. 636)
- geometric series (p. 643)
- inductive hypothesis (p. 670)
- mathematical induction (p. 670)
- recursive formula (p. 658)

Real-World Link
Chambered Nautilus: The spiral formed by the sections of the shell of a chambered nautilus are related to the Fibonacci sequence. The Fibonacci sequence appears in many objects naturally.

Foldables Study Organizer
Sequences and Series: Make this Foldable to help you organize your notes. Begin with one sheet of 11” by 17” paper and four sheets of notebook paper.

1. Fold the short sides of the 11” by 17” paper to meet in the middle.

2. Fold the notebook paper in half lengthwise. Insert two sheets of notebook paper under each tab and staple the edges. Take notes under the appropriate tabs.
## Diagnose Readiness

You have two options for checking Prerequisite Skills.

### Option 1

Take the Quick Check below. Refer to the Quick Review for help.

#### QUICK Check

<table>
<thead>
<tr>
<th>Solve each equation. (Lesson 1-3)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. (-40 = 10 + 5x)</td>
</tr>
<tr>
<td>2. (162 = 2x^4)</td>
</tr>
<tr>
<td>3. (12 - 3x = 27)</td>
</tr>
<tr>
<td>4. (3x^3 + 4 = -20)</td>
</tr>
</tbody>
</table>

#### QUICK Review

<table>
<thead>
<tr>
<th>EXAMPLE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solve the equation (14 = 2x^3 + 700).</td>
</tr>
<tr>
<td>(-686 = 2x^3) Subtract 700 from each side.</td>
</tr>
<tr>
<td>(-343 = x^3) Divide each side by 2.</td>
</tr>
<tr>
<td>(\sqrt[3]{-343} = \sqrt[3]{x^3}) Take the cube root of each side.</td>
</tr>
<tr>
<td>(-7 = x) Simplify.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EXAMPLE 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graph the function ({(1, 1), (2, 1\frac{1}{2}), (3, 1\frac{2}{3}), (4, 1\frac{3}{4}), (5, 1\frac{5}{6})}).</td>
</tr>
<tr>
<td>The domain of a function is the set of all possible x-values.</td>
</tr>
<tr>
<td>So, the domain of this function is ({1, 2, 3, 4, 5}).</td>
</tr>
<tr>
<td>The range of a function is the set of all possible y-values.</td>
</tr>
<tr>
<td>So, the range of this function is ({1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{5}}).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>EXAMPLE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Evaluate the expression (2^m + k + b) if (m = 4, k = -5,) and (b = 1).</td>
</tr>
<tr>
<td>(2^4 + (-5) + 1) Substitute.</td>
</tr>
<tr>
<td>(= 16 - 5 + 1) Simplify.</td>
</tr>
<tr>
<td>(= 2) Zero Exponent Rule</td>
</tr>
</tbody>
</table>

### Option 2

Take the Online Readiness Quiz at [algebra2.com](http://algebra2.com).

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**Chapter 11 Get Ready For Chapter 11**

**GET READY for Chapter 11**

**Diagnose Readiness** You have two options for checking Prerequisite Skills.
Sequences

The numbers in a sequence may not be ordered. For example, the numbers 84, 102, 97, 72, 93, 84, 87, 92, … are a sequence that represents the number of games won by the Houston Astros each season beginning with 1997.

Arithmetic Sequences

The numbers 3, 4, 5, 6, …, representing the number of shingles in each row, are an example of a sequence of numbers. A sequence is a list of numbers in a particular order. Each number in a sequence is called a term. The first term is symbolized by \( a_1 \), the second term is symbolized by \( a_2 \), and so on.

A sequence can also be thought of as a discrete function whose domain is the set of positive integers over some interval.

Many sequences have patterns. For example, in the sequence above for the number of shingles, each term can be found by adding 1 to the previous term. A sequence of this type is called an arithmetic sequence.

An arithmetic sequence is a sequence in which each term after the first is found by adding a constant, called the common difference, to the previous term.

A roofer is nailing shingles to the roof of a house in overlapping rows. There are three shingles in the top row. Since the roof widens from top to bottom, one more shingle is needed in each successive row.

### EXAMPLE

**Find the Next Terms**

Find the next four terms of the arithmetic sequence 55, 49, 43, … .

Find the common difference \( d \) by subtracting two consecutive terms.

\[
49 - 55 = -6 \quad \text{and} \quad 43 - 49 = -6 \quad \text{So,} \quad d = -6.
\]

Now add \(-6\) to the third term of the sequence, and then continue adding \(-6\) until the next four terms are found.

\[
43 \quad 37 \quad 31 \quad 25 \quad 19
\]

\[
+ (-6) \quad + (-6) \quad + (-6) \quad + (-6)
\]

The next four terms of the sequence are 37, 31, 25, and 19.

### CHECK Your Progress 1.

Find the next four terms of the arithmetic sequence \(-1.6, -0.7, 0.2, \ldots \).
It is possible to develop a formula for each term of an arithmetic sequence in terms of the first term \(a_1\) and the common difference \(d\). Consider the sequence in Example 1.

<table>
<thead>
<tr>
<th>Sequence</th>
<th>numbers</th>
<th>55</th>
<th>49</th>
<th>43</th>
<th>37</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>symbols</td>
<td>(a_1)</td>
<td>(a_2)</td>
<td>(a_3)</td>
<td>(a_4)</td>
<td>...</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expressed in Terms of (d) and the First Term</th>
<th>numbers</th>
<th>55 + 0(−6)</th>
<th>55 + 1(−6)</th>
<th>55 + 2(−6)</th>
<th>55 + 3(−6)</th>
<th>...</th>
<th>55 + ((n - 1)(−6))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>symbols</td>
<td>(a_1 + 0 \cdot d)</td>
<td>(a_1 + 1 \cdot d)</td>
<td>(a_1 + 2 \cdot d)</td>
<td>(a_1 + 3 \cdot d)</td>
<td>...</td>
<td>(a_1 + (n-1)d)</td>
</tr>
</tbody>
</table>

The following formula generalizes this pattern for any arithmetic sequence.

\[
 a_n = a_1 + (n - 1)d
\]

You can use the formula to find a term in a sequence given the first term and the common difference or given the first term and some successive terms.

**Real-World Example**

**Find a Particular Term**

**Construction** The table at the right shows typical costs for a construction company to rent a crane for one, two, three, or four months. If the sequence continues, how much would it cost to rent the crane for twelve months?

<table>
<thead>
<tr>
<th>Months</th>
<th>Cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>75,000</td>
</tr>
<tr>
<td>2</td>
<td>90,000</td>
</tr>
<tr>
<td>3</td>
<td>105,000</td>
</tr>
<tr>
<td>4</td>
<td>120,000</td>
</tr>
</tbody>
</table>

**Explore** Since the difference between any two successive costs is $15,000, the costs form an arithmetic sequence with common difference 15,000.

**Plan** You can use the formula for the \(n\)th term of an arithmetic sequence with \(a_1 = 75,000\) and \(d = 15,000\) to find \(a_{12}\), the cost for twelve months.

**Solve**

\[
a_n = a_1 + (n - 1)d
\]

\[
a_{12} = 75,000 + (12 - 1)15,000 \quad n = 12, \quad a_1 = 75,000, \quad d = 15,000
\]

\[
a_{12} = 240,000 \quad \text{Simplify.}
\]

It would cost $240,000 to rent the crane for twelve months.

**Check** You can find terms of the sequence by adding 15,000. \(a_2\) through \(a_{12}\) are 135,000, 150,000, 165,000, 180,000, 195,000, 210,000, 225,000, and 240,000. Therefore, $240,000 is correct.

2. The construction company has a budget of $350,000 for crane rental. The job is expected to last 18 months. Will the company be able to afford the crane rental for the entire job? Explain.
If you are given some of the terms of a sequence, you can use the formula for the \( n \)th term of a sequence to write an equation to help you find the \( n \)th term.

**EXAMPLE**

**Write an Equation for the \( n \)th Term**

Write an equation for the \( n \)th term of the arithmetic sequence

\[ 8, 17, 26, 35, \ldots \]

In this sequence, \( a_1 = 8 \) and \( d = 9 \). Use the \( n \)th term formula to write an equation.

\[
\begin{align*}
  a_n &= a_1 + (n - 1)d \\
  a_n &= 8 + (n - 1)9 \\
  a_n &= 8 + 9n - 9 \\
  a_n &= 9n - 1
\end{align*}
\]

An equation is \( a_n = 9n - 1 \).

**CHECK Your Progress**

3. Write an equation for the \( n \)th term of the arithmetic sequence \(-1.5, -3.5, -5.5, \ldots \).

**ALGEBRA LAB**

**Arithmetic Sequences**

Study the figures below. The length of an edge of each cube is 1 centimeter.

**MODEL AND ANALYZE**

1. Based on the pattern, draw the fourth figure on a piece of isometric dot paper.

2. Find the volumes of the four figures.

3. Suppose the number of cubes in the pattern continues. Write an equation that gives the volume of Figure \( n \).

4. What would the volume of the twelfth figure be?

**Arithmetic Means** Sometimes you are given two terms of a sequence, but they are not successive terms of that sequence. The terms between any two non-successive terms of an arithmetic sequence are called **arithmetic means**.

In the sequence below, 41, 52, and 63 are the three arithmetic means between 30 and 74.

\[ 19, 30, 41, 52, 63, \underline{74}, 85, 96, \ldots \]

The formula for the \( n \)th term of a sequence can be used to find arithmetic means between given terms of a sequence.
Lesson 11-1
Arithmetic Sequences

Alternate Method
You may prefer this method. The four means will be $16 + d$, $16 + 2d$, $16 + 3d$, and $16 + 4d$. The common difference is $d = 91 - (16 + 4d)$ or $d = 15$.

Find the next four terms of each arithmetic sequence.

1. $12, 16, 20, ...$

2. $3, 1, -1, ...$

Find the first five terms of each arithmetic sequence described.

3. $a_1 = 5, d = 3$

4. $a_1 = 14, d = -2$

5. $a_1 = \frac{1}{2}, d = \frac{1}{4}$

6. $a_1 = 0.5, d = -0.2$

7. Find $a_{13}$ for the arithmetic sequence $-17, -12, -7, ...$

Find the indicated term of each arithmetic sequence.

8. $a_1 = 3, d = -5, n = 24$

9. $a_1 = -5, d = 7, n = 13$

10. $a_1 = -4, d = \frac{1}{3}, n = 8$

11. $a_1 = 6.6, d = 1.05, n = 32$

12. ENTERTAINMENT  A basketball team has a halftime promotion where a fan gets to shoot a 3-pointer to try to win a jackpot. The jackpot starts at $5000 for the first game and increases $500 each time there is no winner. Ellis has tickets to the fifteenth game of the season. How much will the jackpot be for that game if no one wins by then?

13. Write an equation for the $n$th term of the arithmetic sequence $-26, -15, -4, 7, ...$

14. Complete: $68$ is the ___th term of the arithmetic sequence $-2, 3, 8, ...$

15. Find the three arithmetic means between $44$ and $92$.

16. Find the three arithmetic means between $2.5$ and $12.5$. 

Example 1 (p. 622)

Find the four arithmetic means between $16$ and $91$.

You can use the $n$th term formula to find the common difference. In the sequence $16, ? , ? , ? , ? , 91, ...$, $a_1$ is $16$ and $a_6$ is $91$.

\[ a_n = a_1 + (n - 1)d \] Formula for the $n$th term
\[ a_6 = 16 + (6 - 1)d \] \( n = 6, a_1 = 16 \)
\[ 91 = 16 + 5d \] \( a_6 = 91 \)
\[ 75 = 5d \] Subtract $16$ from each side.
\[ 15 = d \] Divide each side by $5$.

Now use the value of $d$ to find the four arithmetic means.

$16 \quad 31 \quad 46 \quad 61 \quad 76$

$+15 \quad +15 \quad +15 \quad +15$

The arithmetic means are $31, 46, 61,$ and $76$. CHECK: $76 + 15 = 91 \checkmark$

Example 2 (p. 623)

Example 3 (p. 624)

Example 4 (p. 625)
Find the next four terms of each arithmetic sequence.

17. 9, 16, 23, ...
18. 31, 24, 17, ...
19. −6, −2, 2, ...
20. −8, −5, −2, ...

Find the first five terms of each arithmetic sequence described.

21. \(a_1 = 2, \ d = 13\)
22. \(a_1 = 41, \ d = 5\)
23. \(a_1 = 6, \ d = -4\)
24. \(a_1 = 12, \ d = -3\)

25. Find \(a_8\) if \(a_n = 4 + 3n\).
26. If \(a_n = 1 - 5n\), what is \(a_{10}\)?

Find the indicated term of each arithmetic sequence.

27. \(a_1 = 3, \ d = 7, \ n = 14\)
28. \(a_1 = -4, \ d = -9, \ n = 20\)
29. \(a_1 = 35, \ d = 3, \ n = 101\)
30. \(a_1 = 20, \ d = 4, \ n = 81\)
31. \(a_{12}\) for \(-17, -13, -9, ...\)
32. \(a_{12}\) for \(8, 3, -2, ...\)

33. **TOWER OF PISA** To prove that objects of different weights fall at the same rate, Galileo dropped two objects with different weights from the Leaning Tower of Pisa in Italy. The objects hit the ground at the same time. When an object is dropped from a tall building, it falls about 16 feet in the first second, 48 feet in the second second, and 80 feet in the third second, regardless of its weight. How many feet would an object fall in the sixth second?

34. **GEOLOGY** Geologists estimate that the continents of Europe and North America are drifting apart at a rate of an average of 12 miles every 1 million years, or about 0.75 inch per year. If the continents continue to drift apart at that rate, how many inches will they drift in 50 years? (Hint: \(a_1 = 0.75\))

Complete the statement for each arithmetic sequence.

35. 170 is the ___ term of −4, 2, 8, ...
36. 124 is the ___ term of −2, 5, 12, ...

Write an equation for the \(n\)th term of each arithmetic sequence.

37. 7, 16, 25, 34, ...
38. 18, 11, 4, −3, ...
39. −3, −5, −7, −9, ...
40. −4, 1, 6, 11, ...

Find the arithmetic means in each sequence.

41. 55, __, __, __, __, 115
42. 10, __, __, __, −8
43. −8, __, __, __, __, __, 7
44. 3, __, __, __, __, __, __, 27

Find the next four terms of each arithmetic sequence.

45. \(\frac{1}{3}, \ 1, \ \frac{5}{3}, ...\)
46. \(\frac{18}{5}, \ \frac{16}{5}, \ \frac{14}{5}, ...\)
47. 6.7, 6.3, 5.9, ...
48. 1.3, 3.8, 6.3, ...

Find the first five terms of each arithmetic sequence described.

49. \(a_1 = \frac{4}{3}, \ d = -\frac{1}{3}\)
50. \(a_1 = \frac{5}{8}, \ d = \frac{3}{8}\)
51. **VACATION DAYS** Kono’s employer gives him 1.5 vacation days for each month he works. If Kono has 11 days at the end of one year and takes no vacation time during the next year, how many days will he have at the end of that year?

52. **DRIVING** Olivia was driving her car at a speed of 65 miles per hour. To exit the highway, she began decelerating at a rate of 5 mph per second. How long did it take Olivia to come to a stop?

**SEATING** For Exercises 53–55, use the following information.
The rectangular tables in a reception hall are often placed end-to-end to form one long table. The diagrams below show the number of people who can sit at each of the table arrangements.

53. Make drawings to find the next three numbers as tables are added one at a time to the arrangement.

54. Write an equation representing the \( n \)th number in this pattern.

55. Is it possible to have seating for exactly 100 people with such an arrangement? Explain.

Find the indicated term of each arithmetic sequence.

56. \( a_1 = 5, \ d = \frac{1}{3}, \ n = 12 \)

57. \( a_1 = \frac{5}{2}, \ d = -\frac{3}{2}, \ n = 11 \)

58. \( a_{21} \) for 121, 118, 115, ...

59. \( a_{43} \) for 5, 9, 13, 17, ...

Use the given information to write an equation that represents the \( n \)th number in each arithmetic sequence.

60. The 15th term of the sequence is 66. The common difference is 4.

61. The 100th term of the sequence is 100. The common difference is 7.

62. The tenth term of the sequence is 84. The 21st term of the sequence is 161.

63. The 63rd term of the sequence is 237. The 90th term of the sequence is 75.

64. The 18th term of a sequence is 367. The 30th term of the sequence is 499. How many terms of this sequence are less than 1000?

65. OPEN ENDED Write a real-life application that can be described by an arithmetic sequence with common difference \(-5\).

66. REASONING Explain why the sequence 4, 5, 7, 10, 14, ... is not arithmetic.

67. CHALLENGE The numbers \( x, y, \) and \( z \) are the first three terms of an arithmetic sequence. Express \( z \) in terms of \( x \) and \( y \).

68. Writing in Math Use the information on pages 622 and 623 to explain the relationship between \( n \) and \( a_n \) in the formula for the \( n \)th term of an arithmetic sequence. If \( n \) is the independent variable and \( a_n \) is the dependent variable, what kind of equation relates \( n \) and \( a_n \)? Explain what \( a_1 \) and \( d \) mean in the context of the graph.
69. ACT/SAT What is the first term in the arithmetic sequence?

$$3, 7, 5, 1, 8, ...$$

A 3  
B 9  
C 10  
D 11

70. REVIEW The figures below show a pattern of filled circles and white circles that can be described by a relationship between 2 variables.

Which rule relates \( w \), the number of white circles, to \( f \), the number of dark circles?

F \( w = 3f \)  
H \( w = 2f + 1 \)  
G \( f = \frac{1}{2}w - 1 \)  
J \( f = \frac{1}{3}w \)

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**Spiral Review**

Find the exact solution(s) of each system of equations. (Lesson 10-7)

71. \( x^2 + 2y^2 = 33 \)  
\( x^2 + y^2 - 19 = 2x \)

72. \( x^2 + 2y^2 = 33 \)  
\( x^2 - y^2 = 9 \)

Write each equation in standard form. State whether the graph of the equation is a parabola, circle, ellipse, or hyperbola. Then graph the equation. (Lesson 10-6)

73. \( y^2 - 3x + 6y + 12 = 0 \)

74. \( x^2 - 14x + 4 = 9y^2 - 36y \)

75. If \( y \) varies directly as \( x \) and \( y = 5 \) when \( x = 2 \), find \( y \) when \( x = 6 \). (Lesson 8-4)

Simplify each expression. (Lesson 8-1)

76. \( \frac{39a^3b^4}{13a^b^3} \)

77. \( \frac{k + 3}{5k} \cdot \frac{10k}{k + 3} \)

78. \( \frac{5y - 15z}{42x^2} \div \frac{y - 3z}{14x} \)

Find all the zeros of each function. (Lesson 6-8)

79. \( f(x) = 8x^3 - 36x^2 + 22x + 21 \)

80. \( g(x) = 12x^4 + 4x^3 - 3x^2 - x \)

81. SAVINGS Mackenzie has \$57\ in her bank account. She begins receiving a weekly allowance of \$15\, of which she deposits 20% in her bank account. Write an equation that represents how much money is in Mackenzie’s account after \( x \) weeks. (Lesson 2-4)

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**PREREQUISITE SKILL** Evaluate each expression for the given values of the variable. (Lesson 1-1)

82. \( 3n - 1; n = 1, 2, 3, 4 \)

83. \( 6 - j; j = 1, 2, 3, 4 \)

84. \( 4m + 7; m = 1, 2, 3, 4, 5 \)

85. \( 4 - 2k; k = 3, 4, 5, 6, 7 \)
The numbers of seats in the rows of the amphitheater form an arithmetic sequence. To find the number of people who could sit in the first four rows, add the first four terms of the sequence. That sum is $18 + 22 + 26 + 30$ or 96. A **series** is an indicated sum of the terms of a sequence. Since $18, 22, 26, 30$ is an arithmetic sequence, $18 + 22 + 26 + 30$ is an **arithmetic series**.

$S_n$ represents the sum of the first $n$ terms of a series. For example, $S_4$ is the sum of the first four terms.

To develop a formula for the sum of any arithmetic series, consider the series below.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

Write $S_9$ in two different orders and add the two equations.

$$S_9 = 4 + 11 + 18 + 25 + 32 + 39 + 46 + 53 + 60$$

$$(+) S_9 = 60 + 53 + 46 + 39 + 32 + 25 + 18 + 11 + 4$$

$$2S_9 = 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64 + 64$$

$$2S_9 = 9(64)$$

$$S_9 = \frac{9}{2}(64)$$

Note that the sum had 9 terms.

The sum of the first and last terms of the series is 64.

An arithmetic series $S_n$ has $n$ terms, and the sum of the first and last terms is $a_1 + a_n$. Thus, the formula $S_n = \frac{n}{2}(a_1 + a_n)$ represents the sum of any arithmetic series.
EXAMPLE

Find the Sum of an Arithmetic Series

Find the sum of the first 100 positive integers.

The series is 1 + 2 + 3 + … + 100. Since you can see that \(a_1 = 1\), \(a_{100} = 100\), and \(d = 1\), you can use either sum formula for this series.

**Method 1**

\[
S_n = \frac{n}{2}(a_1 + a_n)
\]

**Method 2**

\[
S_n = \frac{n}{2}[2a_1 + (n - 1)d]
\]

\[
S_{100} = \frac{100}{2}(1 + 100)
\]

\[
S_{100} = 50(101)
\]

\[
S_{100} = 5050
\]

**CHECK Your Progress**

1. Find the sum of the first 50 positive even integers.

**Real-World EXAMPLE**

Find the First Term

**RADIO** A radio station is giving away a total of $124,000 in August. If they increase the amount given away each day by $100, how much should they give away the first day?

You know the values of \(n\), \(S_n\), and \(d\). Use the sum formula that contains \(d\).

\[
S_n = \frac{n}{2}[2a_1 + (n - 1)d]
\]

\[
S_{31} = \frac{31}{2}[2a_1 + (31 - 1)100]
\]

\[
124,000 = \frac{31}{2}(2a_1 + 3000)
\]

\[
8000 = 2a_1 + 3000
\]

\[
5000 = 2a_1
\]

\[
2500 = a_1
\]

The radio station should give away $2500 the first day.

**CHECK Your Progress**

2. **EXERCISE** Aiden did pushups every day in March. He started on March 1st and increased the number of pushups done each day by one. He did a total of 1085 pushups for the month. How many pushups did Aiden do on March 1st?
Sometimes it is necessary to use both a sum formula and the formula for the $n$th term to solve a problem.

**Example**  Find the First Three Terms

Find the first three terms of an arithmetic series in which $a_1 = 9$, $a_n = 105$, and $S_n = 741$.

**Step 1** Since you know $a_1$, $a_n$, and $S_n$, use $S_n = \frac{n}{2}(a_1 + a_n)$ to find $n$.

\[
S_n = \frac{n}{2}(a_1 + a_n) \\
741 = \frac{n}{2}(9 + 105) \\
741 = 57n \\
13 = n
\]

**Step 2** Find $d$.

\[
a_n = a_1 + (n - 1)d \\
105 = 9 + (13 - 1)d \\
96 = 12d \\
8 = d
\]

**Step 3** Use $d$ to determine $a_2$ and $a_3$.

\[
a_2 = 9 + 8 = 17 \\
a_3 = 17 + 8 = 25
\]

The first three terms are 9, 17, and 25.

**Check Your Progress**

3. Find the first three terms of an arithmetic series in which $a_1 = -16$, $a_n = 33$, and $S_n = 68$.

**Sigma Notation** Writing out a series can be time-consuming and lengthy. For convenience, there is a more concise notation called **sigma notation**. The series $3 + 6 + 9 + 12 + \cdots + 30$ can be expressed as $\sum_{n=1}^{10} 3n$. This expression is read the sum of $3n$ as $n$ goes from 1 to 10.

The variable, in this case $n$, is called the **index of summation**.

To generate the terms of a series given in sigma notation, successively replace the index of summation with consecutive integers between the first and last values of the index, inclusive. For the series above, the values of $n$ are 1, 2, 3, and so on, through 10.

There are many ways to represent a given series. If changes are made to the first and last values of the variable and to the formula for the terms of the series, the same terms can be produced. For example, the following expressions produce the same terms.

\[
\sum_{r=4}^{9} (r - 3) \\
\sum_{s=2}^{7} (s - 1) \\
\sum_{j=0}^{5} (j + 1)
\]
EXAMPLE 4

Evaluate \( \sum_{j = 5}^{8} (3j - 4) \).

**Method 1**

Find the terms by replacing \( j \) with 5, 6, 7, and 8. Then add.

\[
\sum_{j = 5}^{8} (3j - 4) = [3(5) - 4] + [3(6) - 4] + [3(7) - 4] + [3(8) - 4]
\]
\[
= 11 + 14 + 17 + 20
\]
\[
= 62
\]

**Method 2**

Since the sum is an arithmetic series, use the formula \( S_n = \frac{n}{2}(a_1 + a_n) \).

There are 4 terms, \( a_1 = 3(5) - 4 \) or 11, and \( a_4 = 3(8) - 4 \) or 20.

\[
S_4 = \frac{4}{2}(11 + 20)
\]
\[
= 62
\]

**CHECK Your Progress**

4. Evaluate \( \sum_{k = 2}^{6} (2k + 1) \).

You can use the sum and sequence features on a graphing calculator to find the sum of a series.

**GrAPhING cAlCuLATOR LAB**

**Sums of Series**

The calculator screen shows the evaluation of \( \sum_{N = 2}^{10} (5N - 2) \). The first four entries for \( \text{seq}( \) are

- the formula for the general term of the series,
- the index of summation,
- the first value of the index, and
- the last value of the index, respectively.

The last entry is always 1 for the types of series that we are considering.

**THINK AND DISCUSS**

1. Explain why you can use any letter for the index of summation.

2. Evaluate \( \sum_{n = 1}^{8} (2n - 1) \) and \( \sum_{j = 5}^{12} (2j - 9) \). Make a conjecture as to their relationship and explain why you think it is true.
Exercises

**HOMEWORK HELP**

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**Example 1**

(p. 630) Find the sum of each arithmetic series.

1. \(5 + 11 + 17 + \cdots + 95\)
2. \(12 + 17 + 22 + \cdots + 102\)
3. \(38 + 35 + 32 + \cdots + 2\)
4. \(101 + 90 + 79 + \cdots + 2\)

5. **TRAINING** To train for a race, Rosmaria runs 1.5 hours longer each week than she did the previous week. In the first week, Rosmaria ran 3 hours. How much time will Rosmaria spend running if she trains for 12 weeks?

**Examples 1, 2**

(p. 630) Find \(S_n\) for each arithmetic series described.

6. \(a_1 = 4, a_n = 100, n = 25\)
7. \(a_1 = 40, n = 20, d = -3\)
8. \(d = -4, n = 21, a_n = 52\)
9. \(d = 5, n = 16, a_n = 72\)

**Example 2**

(p. 630) Find \(a_1\) for each arithmetic series described.

10. \(d = 8, n = 19, S_{19} = 1786\)
11. \(d = -2, n = 12, S_{12} = 96\)

**Example 3**

(p. 631) Find the first three terms of each arithmetic series described.

12. \(a_1 = 11, a_n = 110, S_n = 726\)
13. \(n = 8, a_n = 36, S_n = 120\)

**Example 4**

(p. 632) Find the sum of each arithmetic series.

14. \(\sum_{n=1}^{7} (2n + 1)\)
15. \(\sum_{k=3}^{7} (3k + 4)\)

**Exercises**

Find \(S_n\) for each arithmetic series described.

16. \(a_1 = 7, a_n = 79, n = 8\)
17. \(a_1 = 58, a_n = -7, n = 26\)
18. \(a_1 = 7, d = -2, n = 9\)
19. \(a_1 = 3, d = -4, n = 8\)
20. \(a_1 = 5, d = \frac{1}{2}, n = 13\)
21. \(a_1 = 12, d = \frac{1}{3}, n = 13\)
22. \(d = -3, n = 21, a_n = -64\)
23. \(d = 7, n = 18, a_n = 72\)
24. **TOYS** Jamila is making a wall with building blocks. The top row has one block, the second row has three, the third has five, and so on. How many rows can she make with a set of 100 blocks?

25. **CONSTRUCTION** A construction company will be fined for each day it is late completing a bridge. The daily fine will be $4000 for the first day and will increase by $1000 each day. Based on its budget, the company can only afford $60,000 in total fines. What is the maximum number of days it can be late?

Find \(a_1\) for each arithmetic series described.

26. \(d = 3.5, n = 20, S_{20} = 1005\)
27. \(d = -4, n = 42, S_{42} = -3360\)
28. \(d = 0.5, n = 31, S_{31} = 573.5\)
29. \(d = -2, n = 18, S_{18} = 18\)

Find the first three terms of each arithmetic series described.

30. \(a_1 = 17, a_n = 197, S_n = 2247\)
31. \(a_1 = -13, a_n = 427, S_n = 18,423\)
32. \(n = 31, a_n = 78, S_n = 1023\)
33. \(n = 19, a_n = 103, S_n = 1102\)

**Extra Practice**

See pages 914, 936.

Math Online

Self-Check Quiz at algebra2.com

Lesson 11-2 Arithmetic Series 633
Find the sum of each arithmetic series.

34. \(6 + 13 + 20 + 27 + \cdots + 97\) 
35. \(7 + 14 + 21 + 28 + \cdots + 98\) 
36. \(34 + 30 + 26 + \cdots + 2\) 
37. \(16 + 10 + 4 + \cdots + (-50)\) 
38. \(\sum_{n=1}^{6} (2n + 11)\) 
39. \(\sum_{n=1}^{5} (2 - 3n)\) 
40. \(\sum_{k=7}^{11} (42 - 9k)\) 
41. \(\sum_{i=19}^{23} (5t - 3)\) 
42. \(\sum_{n=1}^{300} (7n - 3)\) 
43. \(\sum_{k=1}^{150} (11 + 2k)\) 

Find \(S_n\) for each arithmetic series described.

44. \(a_1 = 43, \ n = 19, \ a_n = 115\) 
45. \(a_1 = 76, \ n = 21, \ a_n = 176\) 
46. \(a_1 = 91, \ d = -4, \ a_n = 15\) 
47. \(a_1 = -2, \ d = \frac{1}{3}, \ a_n = 9\) 
48. \(d = \frac{1}{5}, \ n = 10, \ a_n = \frac{23}{10}\) 
49. \(d = -\frac{1}{4}, \ n = 20, \ a_n = -\frac{53}{12}\) 
50. Find the sum of the first 1000 positive even integers. 
51. What is the sum of the multiples of 3 between 3 and 999, inclusive? 

52. **AEROSPACE** On the Moon, a falling object falls just 2.65 feet in the first second after being dropped. Each second it falls 5.3 feet farther than it did the previous second. How far would an object fall in the first ten seconds after being dropped? 

53. **SALARY** Mr. Vacarro’s salary this year is $41,000. If he gets a raise of $2500 each year, how much will Mr. Vacarro earn in ten years? 

Use a graphing calculator to find the sum of each arithmetic series.

54. \(\sum_{n=21}^{75} (2n + 5)\) 
55. \(\sum_{n=10}^{50} (3n - 1)\) 
56. \(\sum_{n=20}^{60} (4n + 3)\) 
57. \(\sum_{n=17}^{90} (1.5n + 13)\) 
58. \(\sum_{n=22}^{64} (-n + 70)\) 
59. \(\sum_{n=26}^{50} (-2n + 100)\) 

**OPEN ENDED** Write an arithmetic series for which \(S_5 = 10\). 

**CHALLENGE** State whether each statement is true or false. Explain your reasoning.

61. Doubling each term in an arithmetic series will double the sum. 
62. Doubling the number of terms in an arithmetic series, but keeping the first term and common difference the same, will double the sum. 

**Writing in Math** Use the information on page 629 to explain how arithmetic series apply to amphitheaters. Explain what the sequence and the series that can be formed from the given numbers represent, and show two ways to find the seating capacity of the amphitheater if it has ten rows of seats.
64. ACT/SAT  The measures of the angles of a triangle form an arithmetic sequence. If the measure of the smallest angle is $36^\circ$, what is the measure of the largest angle?

A $75^\circ$  B $84^\circ$  C $90^\circ$  D $97^\circ$

65. REVIEW  How many 5-inch cubes can be placed completely inside a box that is 10 inches long, 15 inches wide, and 5 inches tall?

F 5  H 20  G 6  J 15

---

### Spiral Review

Find the indicated term of each arithmetic sequence. (Lesson 11-1)

66. $a_1 = 46, d = 5, n = 14$  
67. $a_1 = 12, d = -7, n = 22$

Solve each system of inequalities by graphing. (Lesson 10-7)

68. $9x^2 + y^2 < 81$  
69. $(y - 3)^2 \geq x + 2$

$x^2 + y^2 \geq 16$  
$x^2 \leq y + 4$

Write an equivalent logarithmic equation. (Lesson 9-2)

70. $5^x = 45$  
71. $7^3 = x$  
72. $b^y = x$

73. PAINTING  Two employees of a painting company paint houses together. One painter can paint a house alone in 3 days, and the other painter can paint the same size house alone in 4 days. How long will it take them to paint one house if they work together? (Lesson 8-6)

Simplify. (Lesson 7-5)

74. $5\sqrt{3} - 4\sqrt{3}$  
75. $\sqrt{26} \cdot \sqrt{39} \cdot \sqrt{14}$  
76. $(\sqrt{10} - \sqrt{6})(\sqrt{5} + \sqrt{3})$

Solve each equation by completing the square. (Lesson 5-5)

77. $x^2 + 9x + 20.25 = 0$  
78. $9x^2 + 96x + 256 = 0$  
79. $x^2 - 3x - 20 = 0$

Use a graphing calculator to find the value of each determinant. (Lesson 4-5)

80. $\begin{vmatrix} 1.3 & 7.2 \\ 6.1 & 5.4 \end{vmatrix}$  
81. $\begin{vmatrix} 6.1 & 4.8 \\ 9.7 & 3.5 \end{vmatrix}$  
82. $\begin{vmatrix} 8 & 6 & -5 \\ 10 & -7 & 3 \\ 9 & 14 & -6 \end{vmatrix}$

Solve each system of equations by using either substitution or elimination. (Lesson 3-2)

83. $a + 4b = 6$  
$3a + 2b = -2$

84. $10x - y = 13$  
$3x - 4y = 15$

85. $3c - 7d = -1$  
$2c - 6d = -6$

---

PREREQUISITE SKILL  Evaluate the expression $a \cdot b^{n-1}$ for the given values of $a$, $b$, and $n$. (Lesson 1-1)

86. $a = 1, b = 2, n = 5$  
87. $a = 2, b = -3, n = 4$  
88. $a = 18, b = \frac{1}{3}, n = 6$
11-3

Geometric Sequences

Main Ideas
• Use geometric sequences.
• Find geometric means.

New Vocabulary
geometric sequence
common ratio
geometric means

When you drop a ball, it never rebounds to the height from which you dropped it. Suppose a ball is dropped from a height of three feet, and each time it falls, it rebounds to 60% of the height from which it fell. The heights of the ball’s rebounds form a sequence.

Geometric Sequences
The height of the first rebound of the ball is $3(0.6)$ or 1.8 feet. The height of the second rebound is $1.8(0.6)$ or 1.08 feet. The height of the third rebound is $1.08(0.6)$ or 0.648 feet. The sequence of heights is an example of a geometric sequence. A geometric sequence is a sequence in which each term after the first is found by multiplying the previous term by a nonzero constant $r$ called the common ratio. As with an arithmetic sequence, you can label the terms of a geometric sequence as $a_1, a_2, a_3$, and so on, $a_1 \neq 0$. The $n$th term is $a_n$ and the previous term is $a_{n-1}$. So, $a_n = r(a_{n-1})$. Thus, $r = \frac{a_n}{a_{n-1}}$. That is, the common ratio can be found by dividing any term by its previous term.

Standardized Test Example

What is the missing term in the geometric sequence: 8, 20, 50, 125, ____?

A  75
B  200
C  250
D  312.5

Read the Test Item

Since $\frac{20}{8} = 2.5$, $\frac{50}{20} = 2.5$, and $\frac{125}{50} = 2.5$, the common ratio is 2.5.

Solve the Test Item

To find the missing term, multiply the last given term by 2.5: $125(2.5) = 312.5$. The answer is D.

Find the Next Term

1. What is the missing term in the geometric sequence: −120, 60, −30, 15, ____?

F  −7.5
G  0
H  7.5
J  10
You have seen that each term of a geometric sequence after the first term can be expressed in terms of $r$ and its previous term. It is also possible to develop a formula that expresses each term of a geometric sequence in terms of $r$ and the first term $a_1$. Study the patterns in the table for the sequence 2, 6, 18, 54, … .

<table>
<thead>
<tr>
<th>Sequence</th>
<th>numbers</th>
<th>2</th>
<th>6</th>
<th>18</th>
<th>54</th>
<th>…</th>
</tr>
</thead>
<tbody>
<tr>
<td>symbols</td>
<td>$a_1$</td>
<td>$a_2$</td>
<td>$a_3$</td>
<td>$a_4$</td>
<td>…</td>
<td>$a_n$</td>
</tr>
</tbody>
</table>

**Expressed in Terms of $r$ and the Previous Term**

| numbers | 2 | 2(3) | 6(3) | 18(3) | … |
| symbols | $a_1$ | $a_1 \cdot r$ | $a_2 \cdot r$ | $a_3 \cdot r$ | … | $a_n \cdot r$ |

| numbers | 2 | 2(3) | 2(9) | 2(27) | … |
| symbols | $2(3^0)$ | $2(3^1)$ | $2(3^2)$ | $2(3^3)$ | … |

| symbols | $a_1 \cdot r^0$ | $a_1 \cdot r^1$ | $a_1 \cdot r^2$ | $a_1 \cdot r^3$ | … | $a_1 \cdot r^n - 1$ |

The three entries in the last column all describe the $n$th term of a geometric sequence. This leads to the following formula.

**KEY CONCEPT**

### $n$th Term of a Geometric Sequence

The $n$th term $a_n$ of a geometric sequence with first term $a_1$ and common ratio $r$ is given by the following formula, where $n$ is any positive integer.

$$a_n = a_1 \cdot r^{n-1}$$

**EXAMPLE**

**Find a Term Given the First Term and the Ratio**

Find the eighth term of a geometric sequence for which $a_1 = -3$ and $r = -2$.

Formula for $n$th term

$$a_n = a_1 \cdot r^{n-1}$$

$n = 8, a_1 = -3, r = -2$

$a_8 = (-3) \cdot (-2)^8 - 1$

$a_8 = (-3) \cdot (-256)$

$a_8 = 384$

Multiply.

**CHECK Your Progress**

2. Find the sixth term of a geometric sequence for which $a_1 = \frac{1}{9}$ and $r = 3$.

**EXAMPLE**

**Write an Equation for the $n$th Term**

Write an equation for the $n$th term of the geometric sequence 3, 12, 48, 192, … .

Formula for $n$th term

$$a_n = a_1 \cdot r^{n-1}$$

$a_1 = 3, r = 4$

$a_n = 3 \cdot 4^{n-1}$

**CHECK Your Progress**

3. Write an equation for the $n$th term of the geometric sequence 18, $-3$, $\frac{1}{2}$, $-\frac{1}{12}$, … .
Alternate Method

You may prefer this method. The three means will be 2.25, 2.25^2, and 2.25^3.
Then the common ratio is \( r = \frac{576}{2.25^3} \), or \( r = \frac{576}{2.25} \). Thus, \( r = 4 \).

Geometric Means

In Lesson 11-1, you learned that missing terms between two nonsuccessive terms in an arithmetic sequence are called arithmetic means. Similarly, the missing terms(s) between two nonsuccessive terms of a geometric sequence are called geometric means. For example, 6, 18, and 54 are three geometric means between 2 and 162 in the sequence 2, 6, 18, 54, 162, … . You can use the common ratio to find the geometric means in a sequence.

EXAMPLE

Find Geometric Means

Find three geometric means between 2.25 and 576.

Use the \( n \)th term formula to find the value of \( r \). In the sequence 2.25, ____ , ____ , 576, \( a_1 \) is 2.25 and \( a_5 \) is 576.

\[
\begin{align*}
a_n &= a_1 \cdot r^{n-1} \quad \text{Formula for } n \text{th term} \\
a_5 &= 2.25 \cdot r^4 \\
576 &= 2.25 \cdot r^4 \\
256 &= 2.25r^4 \\
\pm 4 &= r 
\end{align*}
\]

There are two possible common ratios, so there are two possible sets of geometric means. Use each value of \( r \) to find three geometric means.

\[
\begin{align*}
r &= 4 & r &= -4 \\
a_2 &= 2.25(4) \text{ or } 9 & a_2 &= 2.25(-4) \text{ or } -9 \\
a_3 &= 9(4) \text{ or } 36 & a_3 &= -9(-4) \text{ or } 36 \\
a_4 &= 36(4) \text{ or } 144 & a_4 &= 36(-4) \text{ or } -144 
\end{align*}
\]

The geometric means are 9, 36, and 144, or -9, 36, and -144.

Check Your Progress

5. Find two geometric means between 4 and 13.5.
Lesson 11-3 Geometric Sequences

Example 1 (p. 636)
1. Find the next two terms of the geometric sequence 20, 30, 45, ... .
2. Find the first five terms of the geometric sequence for which $a_1 = -2$ and $r = 3$.

3. **STANDARDIZED TEST PRACTICE** What is the missing term in the geometric sequence: $-\frac{1}{4}, \frac{1}{2}, -1, 2, ____$?
   - A $-4$  
   - B $-\frac{3}{2}$  
   - C $\frac{3}{2}$  
   - D 4

Example 2 (p. 637)
4. Find $a_9$ for the geometric sequence 60, 30, 15, ... .
5. Find $a_8$ for the geometric sequence $\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, ...$.

**Find the indicated term of each geometric sequence.**
6. $a_1 = 7, r = 2, n = 4$  
7. $a_1 = 3, r = \frac{1}{3}, n = 5$

Example 3 (p. 637)
8. Write an equation for the $n$th term of the geometric sequence 4, 8, 16, ... .
9. Write an equation for the $n$th term of the geometric sequence 15, 5, $\frac{5}{3}$, ... .

Example 4 (p. 638)
10. $a_3 = 24, r = \frac{1}{2}, n = 7$
11. $a_3 = 32, r = -0.5, n = 6$

Example 5 (p. 638)
12. Find two geometric means between 1 and 27.
13. Find two geometric means between 2 and 54.

**Exercises**

**Homework**

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</tr>
<tr>
<td>28, 29</td>
<td>3</td>
</tr>
<tr>
<td>30–33</td>
<td>4</td>
</tr>
<tr>
<td>34–37</td>
<td>5</td>
</tr>
</tbody>
</table>

**Find the next two terms of each geometric sequence.**
14. 405, 135, 45, ...
15. 81, 108, 144, ...
16. 16, 24, 36, ...
17. 162, 108, 72, ...

**Find the first five terms of each geometric sequence described.**
18. $a_1 = 2, r = -3$
19. $a_1 = 1, r = 4$
20. Find $a_7$ if $a_1 = 12$ and $r = \frac{1}{2}$.
21. Find $a_6$ if $a_1 = \frac{1}{3}$ and $r = 6$.

**INTEREST** An investment pays interest so that each year the value of the investment increases by 10%. How much is an initial investment of $1000 worth after 5 years?

**SALARIES** Geraldo’s current salary is $40,000 per year. His annual pay raise is always a percent of his salary at the time. What would his salary be if he got four consecutive 4% increases?

**Find the indicated term of each geometric sequence.**
22. $a_1 = \frac{1}{3}, r = 3, n = 8$
23. $a_1 = \frac{1}{64}, r = 4, n = 9$
24. $a_9$ for $a_1 = \frac{1}{5}, 1, 5, ...$
25. $a_7$ for $\frac{1}{32}, \frac{1}{16}, \frac{1}{8}, ...$
26. $a_4 = 16, r = 0.5, n = 8$
27. $a_6 = 3, r = 2, n = 12$
Write an equation for the $n$th term of each geometric sequence.

30. $36, 12, 4, \ldots$  
31. $64, 16, 4, \ldots$  
32. $-2, 10, -50, \ldots$  
33. $4, -12, 36, \ldots$

Find the geometric means in each sequence.

34. $9, ? , ? , ? , 144$  
35. $4, ? , ? , ? , 324$  
36. $-2, 10, -50, \ldots$  

Find the next two terms of each geometric sequence.

38. $5\frac{2}{3}, 5\frac{10}{9}, \ldots$  
39. $\frac{1}{3}, \frac{5}{6}, \frac{25}{12}, \ldots$  
40. $1.25, -1.5, 1.8, \ldots$  
41. $1.4, -3.5, 8.75, \ldots$

Find the first five terms of each geometric sequence described.

42. $a_1 = 243, r = \frac{1}{3}$  
43. $a_1 = 576, r = -\frac{1}{2}$  
44. $\text{ART}$ A one-ton ice sculpture is melting so that it loses one-eighth of its weight per hour. How much of the sculpture will be left after five hours? Write your answer in pounds.

MEDICINE For Exercises 45 and 46, use the following information.

Iodine-131 is a radioactive element used to study the thyroid gland.

45. RESEARCH Use the Internet or other resource to find the half-life of Iodine-131, rounded to the nearest day. This is the amount of time it takes for half of a sample of Iodine-131 to decay into another element.

46. How much of an 80-milligram sample of Iodine-131 would be left after 32 days?

Find the indicated term of each geometric sequence.

47. $a_1 = 16,807, r = \frac{3}{7}, n = 6$  
48. $a_1 = 4096, r = \frac{1}{4}, n = 8$  
49. $a_8$ for $4, -12, 36, \ldots$  
50. $a_6$ for $540, 90, 15, \ldots$  
51. $a_4 = 50, r = 2, n = 8$  
52. $a_4 = 1, r = 3, n = 10$

53. OPEN ENDED Write a geometric sequence with a common ratio of $\frac{2}{3}$.

54. FIND THE ERROR Marika and Lori are finding the seventh term of the geometric sequence $9, 3, 1, \ldots$ . Who is correct? Explain your reasoning.

Marika

$r = \frac{3}{9} \text{ or } \frac{1}{3}$

$a_7 = 9 \left( \frac{1}{3} \right)^{7-1} = \frac{1}{81}$

Lori

$r = \frac{9}{3} \text{ or } 3$

$a_7 = 9 \cdot 3^{7-1} = 6561$

55. Which One Doesn’t Belong? Identify the sequence that does not belong with the other three. Explain your reasoning.

1, 4, 16, ...  
3, 9, 27, ...  
9, 16, 25, ...  
$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$
CHALLENGE  Determine whether each statement is true or false. If true, explain. If false, provide a counterexample.

56. Every sequence is either arithmetic or geometric.
57. There is no sequence that is both arithmetic and geometric.

58. Writing in Math  Use the information on pages 636 and 637 to explain the relationship between \( n \) and \( a_n \) in the formula for the \( n \)th term of a geometric sequence. If \( n \) is the independent variable and \( a_n \) is the dependent variable, what kind of equation relates \( n \) and \( a_n \)? Explain what \( r \) represents in the context of the relationship.

59. ACT/SAT  The first four terms of a geometric sequence are shown in the table. What is the tenth term in the sequence?

\[
\begin{array}{c|c}
\hline
n & a_n \\
\hline
1 & 144 \\
2 & 72 \\
3 & 36 \\
4 & 18 \\
\hline
\end{array}
\]

A  0  
B  \( \frac{9}{64} \)  
C  \( \frac{9}{32} \)  
D  \( \frac{9}{16} \)

60. REVIEW  The table shows the cost of jelly beans depending on the amount purchased. Which conclusion can be made based on the table?

<table>
<thead>
<tr>
<th>Number of Pounds</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$14.95</td>
</tr>
<tr>
<td>20</td>
<td>$57.80</td>
</tr>
<tr>
<td>50</td>
<td>$139.50</td>
</tr>
<tr>
<td>100</td>
<td>$269.00</td>
</tr>
</tbody>
</table>

F  The cost of 10 pounds of jelly beans would be more than $30.
G  The cost of 200 pounds of jelly beans would be less than $540.
H  The cost of jelly beans is always more than $2.70 per pound.
J  The cost of jelly beans is always less than $2.97 per pound.

Spiral Review

Find \( S_n \) for each arithmetic series described.  \( \text{(Lesson 11-2)} \)

61. \( a_1 = 11, a_n = 44, n = 23 \)
62. \( a_1 = -5, d = 3, n = 14 \)

Find the arithmetic means in each sequence.  \( \text{(Lesson 11-1)} \)

63. 15, \( \_ \), \( \_ \), 27
64. \(-8, \_ \), \( \_ \), \( \_ \), \( \_ \), \(-24 \)

65. GEOMETRY  Find the perimeter of a triangle with vertices at (2, 4), (-1, 3) and (1, -3).  \( \text{(Lesson 10-1)} \)

GET READY for the Next Lesson

PREREQUISITE SKILL  Evaluate each expression.  \( \text{(Lesson 1-1)} \)

66. \( \frac{1 - 2^7}{1 - 2} \)
67. \( \frac{1 - \left(\frac{1}{2}\right)^6}{1 - \frac{1}{2}} \)
68. \( \frac{1 - \left(-\frac{1}{3}\right)^5}{1 - \left(-\frac{1}{3}\right)} \)
Graphing Calculator Lab

Limits

You may have noticed that in some geometric sequences, the later the term in the sequence, the closer the value is to 0. Another way to describe this is that as \( n \) increases, \( a_n \) approaches 0. The value that the terms of a sequence approach, in this case 0, is called the limit of the sequence. Other types of infinite sequences may also have limits. If the terms of a sequence do not approach a unique value, we say that the limit of the sequence does not exist.

**Activity 1** Find the limit of the geometric sequence \( 1, \frac{1}{3}, \frac{1}{9}, \ldots \).

**Step 1** Enter the sequence.
- The formula for this sequence is \( a_n = \left(\frac{1}{3}\right)^{n-1} \).
- Position the cursor on \( L1 \) in the **STAT EDIT** Edit … screen and enter the formula \( \text{seq}(N,N,1,10,1) \). This generates the values 1, 2, \( \ldots \), 10 of the index \( N \).
- Position the cursor on \( L2 \) and enter the formula \( \text{seq}((1/3)^{(N-1)},N,1,10,1) \). This generates the first ten terms of the sequence.

**KEystrokes:** Review sequences in the Graphing Calculator Lab on page 632.

Notice that as \( n \) increases, the terms of the given sequence get closer and closer to 0. If you scroll down, you can see that for \( n \geq 8 \) the terms are so close to 0 that the calculator expresses them in scientific notation. This suggests that the limit of the sequence is 0.

**Step 2** Graph the sequence.
- Use a **STAT PLOT** to graph the sequence. Use \( L1 \) as the Xlist and \( L2 \) as the Ylist.

**KEystrokes:** Review **STAT PLOTS** on page 92.

The graph also shows that, as \( n \) increases, the terms approach 0. In fact, for \( n \geq 6 \), the marks appear to lie on the horizontal axis. This strongly suggests that the limit of the sequence is 0.

**Exercises**

Use a graphing calculator to find the limit, if it exists, of each sequence.

1. \( a_n = \left(\frac{1}{2}\right)^n \)
2. \( a_n = \left(-\frac{1}{2}\right)^n \)
3. \( a_n = 4^n \)
4. \( a_n = \frac{1}{n^2} \)
5. \( a_n = \frac{2^n}{2^n + 1} \)
6. \( a_n = \frac{n^2}{n + 1} \)
Suppose you e-mail a joke to three friends on Monday. Each of those friends sends the joke on to three of their friends on Tuesday. Each person who receives the joke on Tuesday sends it to three more people on Wednesday, and so on.

**Geometric Series** Notice that every day, the number of people who read your joke is three times the number that read it the day before. By Sunday, the number of people, including yourself, who have read the joke is $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, or 3280!

The numbers 1, 3, 9, 27, 81, 243, 729, and 2187 form a geometric sequence in which $a_1 = 1$ and $r = 3$. The indicated sum of the numbers in the sequence, $1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187$, is called a **geometric series**.

To develop a formula for the sum of a geometric series, consider the series given in the e-mail situation above. Multiply each term in the series by the common ratio and subtract the result from the original series.

\[
S_8 = 1 + 3 + 9 + 27 + 81 + 243 + 729 + 2187
\]
\[
(-) 3S_8 = 3 + 9 + 27 + 81 + 243 + 729 + 2187 + 6561
\]
\[
(1 - 3)S_8 = 1 + 0 + 0 + 0 + 0 + 0 + 0 + 0 - 6561
\]

\[
S_8 = \frac{1 - 6561}{1 - 3} = \frac{-6560}{-2} = 3280
\]

A rational expression like this can be used to find the sum of any geometric series.
The sum $S_n$ of the first $n$ terms of a geometric series is given by

$$S_n = \frac{a_1 - ar^n}{1 - r} \quad \text{or} \quad S_n = \frac{a_1(1 - r^n)}{1 - r}, \text{ where } r \neq 1.$$ 

You cannot use the formula for the sum with a geometric series for which $r = 1$ because division by 0 would result. In a geometric series with $r = 1$, the terms are constant. For example, $4 + 4 + 4 + \ldots + 4$ is such a series. In general, the sum of $n$ terms of a geometric series with $r = 1$ is $n \cdot a_1$.

**Real-World Link**

The development of vaccines for many diseases has helped to prevent infection. Vaccinations are commonly given to children.

**EXAMPLE**

Evaluate a Sum Written in Sigma Notation

Evaluate $\sum_{n=1}^{6} 5 \cdot 2^{n-1}$.

Method 1 Since the sum is a geometric series, you can use the formula.

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$S_6 = \frac{5(1 - 2^6)}{1 - 2} \quad n = 6, a_1 = 5, r = 2$$

$$S_6 = \frac{5(-63)}{-1} = 315$$

Simplify.

**Health** Contagious diseases can spread very quickly. Suppose five people are ill during the first week of an epidemic, and each person who is ill spreads the disease to four people by the end of the next week. By the end of the tenth week of the epidemic, how many people have been affected by the illness?

This is a geometric series with $a_1 = 5$, $r = 4$, and $n = 10$.

$$S_n = \frac{a_1(1 - r^n)}{1 - r} \quad \text{Sum formula}$$

$$S_{15} = \frac{5(1 - 4^{10})}{1 - 4} \quad n = 10, a_1 = 5, r = 4$$

$$S_{15} = 1,747,625 \quad \text{Use a calculator.}$$

After ten weeks, 1,747,625 people have been affected by the illness.

**Check Your Progress**

1. **Games** Maria arranges some rows of dominoes so that after she knocks over the first one, each domino knocks over two more dominoes when it falls. If there are ten rows, how many dominoes does Maria use?

You can use sigma notation to represent geometric series.
How can you find the sum of a geometric series if you know the first and last terms and the common ratio, but not the number of terms? You can use the formula for the \( n \)th term of a geometric sequence or series, \( a_n = a_1 \cdot r^{n-1} \), to find an expression involving \( r^n \).

\[
\begin{align*}
a_n &= a_1 \cdot r^{n-1} & \text{Formula for } n \text{th term} \\
a_n \cdot r &= a_1 \cdot r^{n-1} \cdot r & \text{Multiply each side by } r \\
a_n \cdot r &= a_1 \cdot r^n & r^{n-1} \cdot r^1 = r^n = r^{n-1+1} \text{ or } r^n 
\end{align*}
\]

Now substitute \( a_n \cdot r \) for \( a_1 \cdot r^n \) in the formula for the sum of geometric series. The result is \( S_n = \frac{a_1 - a_n r}{1 - r} \).

**Example** Use the Alternate Formula for a Sum

Find the sum of a geometric series for which \( a_1 = 15,625 \), \( a_n = -5 \), and \( r = \frac{-1}{5} \).

Since you do not know the value of \( n \), use the formula derived above.

\[
S_n = \frac{a_1 - a_n r}{1 - r} \quad \text{Alternate sum formula}
\]

\[
= \frac{15,625 - (-5) \left( \frac{-1}{5} \right)}{1 - \left( \frac{-1}{5} \right)} \quad a_1 = 15,625; a_n = -5; r = \frac{-1}{5}
\]

\[
= \frac{15,624}{6} \quad \text{or} \quad 13,020 \quad \text{Simplify.}
\]

**Check Your Progress**

2. Evaluate \( \sum_{n=1}^{5} -4 \cdot 3^{n-1} \).
Specific Terms  You can use the formula for the sum of a geometric series to help find a particular term of the series.

**EXAMPLE**  Find the First Term of a Series

4. Find \( a_1 \) in a geometric series for which \( S_8 = 39,360 \) and \( r = 3 \).

\[
S_n = \frac{a_1(1 - r^n)}{1 - r}
\]

Sum formula

\[
39,360 = \frac{a_1(1 - 3^8)}{1 - 3}
\]

\( S_8 = 39,360; r = 3; n = 8 \)

\[
39,360 = \frac{-6560a_1}{-2}
\]

Subtract.

\[
39,360 = 3280a_1
\]

Divide.

\[
a_1 = 12
\]

Divide each side by 3280.

CHECK Your Progress

4. Find \( a_1 \) in a geometric series for which \( S_7 = 258 \) and \( r = -2 \).

Find \( S_n \) for each geometric series described.

1. \( a_1 = 5, r = 2, n = 14 \)

2. \( a_1 = 243, r = -\frac{2}{3}, n = 5 \)

Find the sum of each geometric series.

3. \( 54 + 36 + 24 + 16 + \cdots \) to 6 terms

4. \( 3 - 6 + 12 - \cdots \) to 7 terms

WEATHER  Heavy rain caused a river to rise. The river rose three inches the first day, and each day it rose twice as much as the previous day. How much did the river rise in five days?

Find the sum of each geometric series.

6. \( \sum_{n=1}^{5} \frac{1}{4} \cdot 2^{n-1} \)

7. \( \sum_{n=1}^{8} \frac{1}{3} \cdot \left(\frac{1}{3}\right)^{n-1} \)

8. \( \sum_{n=1}^{12} \frac{1}{5} (-2)^n \)

9. \( \sum_{n=1}^{8} \frac{1}{3} \cdot 5^{n-1} \)

10. \( \sum_{n=1}^{6} 100 \left(\frac{1}{2}\right)^{n-1} \)

11. \( \sum_{n=1}^{9} \frac{1}{27} (-3)^{n-1} \)

Find \( S_n \) for each geometric series described.

12. \( a_1 = 12, a_5 = 972, r = -3 \)

13. \( a_1 = 3, a_n = 46,875, r = -5 \)

14. \( a_1 = 5, a_n = 81,920, r = 4 \)

15. \( a_1 = -8, a_6 = -256, r = 2 \)

Find the indicated term for each geometric series described.

16. \( S_n = \frac{381}{64}, r = \frac{1}{2}, n = 7; a_1 \)

17. \( S_n = 33, a_n = 48, r = -2; a_1 \)

18. \( S_n = 443, r = \frac{1}{3}, n = 6; a_1 \)

19. \( S_n = -242, a_n = -162, r = 3; a_1 \)
Exercises

28. **GENEALOGY**  In the book *Roots*, author Alex Haley traced his family history back many generations to the time one of his ancestors was brought to America from Africa. If you could trace your family back for 15 generations, starting with your parents, how many ancestors would there be?

29. **LEGENDS**  There is a legend of a king who wanted to reward a boy for a good deed. The king gave the boy a choice. He could have $1,000,000 at once, or he could be rewarded daily for a 30-day month, with one penny on the first day, two pennies on the second day, and so on, receiving twice as many pennies each day as the previous day. How much would the second option be worth?

<table>
<thead>
<tr>
<th>Day</th>
<th>Payment</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1¢</td>
</tr>
<tr>
<td>2</td>
<td>2¢</td>
</tr>
<tr>
<td>3</td>
<td>4¢</td>
</tr>
<tr>
<td>4</td>
<td>8¢</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>30</td>
<td>?</td>
</tr>
<tr>
<td>Total</td>
<td>?</td>
</tr>
</tbody>
</table>

Find the indicated term for each geometric series described.

34. \( S_n = 165, a_n = 48, r = -\frac{2}{3}, a_1 \)
35. \( S_n = 688, a_n = 16, r = -\frac{1}{2}, a_1 \)
36. \( S_n = -364, r = -3, n = 6; a_1 \)
37. \( S_n = 1530, r = 2, n = 8; a_1 \)

Find \( S_n \) for each geometric series described.

38. \( a_1 = 162, r = \frac{1}{3}, n = 6 \)
39. \( a_1 = 80, r = -\frac{1}{2}, n = 7 \)
40. \( a_1 = 625, r = 0.4, n = 8 \)
41. \( a_1 = 4, r = 0.5, n = 8 \)
42. \( a_2 = -36, a_5 = 972, n = 7 \)
43. \( a_3 = -36, a_6 = -972, n = 10 \)
44. \( a_1 = 4, a_n = 236,196, r = 3 \)
45. \( a_1 = 125, a_n = \frac{1}{125}, r = \frac{1}{5} \)

Find the sum of each geometric series.

30. \( 4096 - 512 + 64 - \cdots \) to 5 terms
31. \( 7 + 21 + 63 + \cdots \) to 10 terms
32. \( \sum_{n=1}^{9} 5 \cdot 2^{n-1} \)
33. \( \sum_{n=1}^{6} 2(-3)^{n-1} \)

Find \( S_n \) for each geometric series described.

46. \( \frac{1}{16} + \frac{1}{4} + 1 + \cdots \) to 7 terms
47. \( \frac{1}{9} - \frac{1}{3} + 1 - \cdots \) to 6 terms
48. \( \sum_{n=1}^{8} 64\left(\frac{3}{4}\right)^n \)
49. \( \sum_{n=1}^{20} 3 \cdot 2^{n-1} \)
50. \( \sum_{n=1}^{16} 4 \cdot 3^{n-1} \)
51. \( \sum_{n=1}^{16} 144\left(-\frac{1}{2}\right)^{n-1} \)

Real-World Link

Some of the best-known legends involving a king are the Arthurian legends. According to the legends, King Arthur reigned over Britain before the Saxon conquest. Camelot was the most famous castle in the medieval legends of King Arthur.
Find the indicated term for each geometric series described.

52. \( S_n = 315, r = 0.5, n = 6; a_2 \)

53. \( S_n = 249.92, r = 0.2, n = 5, a_3 \)

54. **WATER TREATMENT** A certain water filtration system can remove 80% of the contaminants each time a sample of water is passed through it. If the same water is passed through the system three times, what percent of the original contaminants will be removed from the water sample?

Use a graphing calculator to find the sum of each geometric series.

55. \( \sum_{n=1}^{20} 3(-2)^n - 1 \)

56. \( \sum_{n=1}^{15} 2\left(\frac{1}{2}\right)^n - 1 \)

57. \( \sum_{n=1}^{10} 5(0.2)^n - 1 \)

58. \( \sum_{n=1}^{13} 6\left(\frac{1}{3}\right)^n - 1 \)

59. **OPEN ENDED** Write a geometric series for which \( r = \frac{1}{2} \) and \( n = 4 \).

60. **REASONING** Explain how to write the series \( 2 + 12 + 72 + 432 + 2592 \) using sigma notation.

61. **CHALLENGE** If \( a_1 \) and \( r \) are integers, explain why the value of \( \frac{a_1 - a_1r^n}{1 - r} \) must also be an integer.

62. **REASONING** Explain, using geometric series, why the polynomial \( 1 + x + x^2 + x^3 \) can be written as \( \frac{x^4 - 1}{x - 1} \), assuming \( x \neq 1 \).

63. **Writing in Math** Use the information on page 643 to explain how e-mailing a joke is related to a geometric series. Include an explanation of how the situation could be changed to make it better to use a formula than to add terms.

**HOT Problems:**

64. **ACT/SAT** The first term of a geometric series is \(-1\), and the common ratio is \(-3\). How many terms are in the series if its sum is 182?
- A 6
- B 7
- C 8
- D 9

65. **REVIEW** Which set of dimensions corresponds to a rectangle similar to the one shown below?
- A 3 units by 1 unit
- B 12 units by 9 units
- C 13 units by 8 units
- D 18 units by 12 units
Find the geometric means in each sequence. (Lesson 11-3)

66. \(\frac{1}{24}, \ ? , \ ? , \ ? , 54\)
67. \(-2, \ ? , \ ? , \ ? , \ ? , \frac{-243}{16}\)

Find the sum of each arithmetic series. (Lesson 11-2)

68. \(50 + 44 + 38 + \ldots + 8\)
69. \(\sum_{n=1}^{12} (2n + 3)\)

Solve each equation. Check your solutions. (Lesson 8–6)

70. \(\frac{1}{y + 1} - \frac{3}{y - 3} = 2\)
71. \(\frac{6}{a - 7} = \frac{a - 49}{a^2 - 7a} + \frac{1}{a}\)

Determine whether each graph represents an odd-degree polynomial function or an even-degree polynomial function. Then state how many real zeros each function has. (Lesson 6-4)

72. \(y\)
73. \(y\)

Factor completely. If the polynomial is not factorable, write \textit{prime}. (Lesson 5-3)

74. \(3d^2 + 2d - 8\)
75. \(42pq - 35p + 18q - 15\)
76. \(13xyz + 3x^2z + 4k\)

**VOTING** For Exercises 77–79, use the table that shows the percent of the Iowa population of voting age that voted in each presidential election from 1984–2004. (Lesson 2-5)

77. Draw a scatter plot in which \(x\) is the number of elections since the 1984 election.

78. Find a linear prediction equation.

79. Predict the percent of the Iowa voting age population that will vote in the 2012 election.

### Year vs. Percent of Voting Age Population Voting in Iowa

<table>
<thead>
<tr>
<th>Year</th>
<th>Percent</th>
</tr>
</thead>
<tbody>
<tr>
<td>1984</td>
<td>76.0</td>
</tr>
<tr>
<td>1988</td>
<td>75.7</td>
</tr>
<tr>
<td>1992</td>
<td>72.4</td>
</tr>
<tr>
<td>1996</td>
<td>80.2</td>
</tr>
<tr>
<td>2000</td>
<td>73.2</td>
</tr>
<tr>
<td>2004</td>
<td>75.0</td>
</tr>
</tbody>
</table>

Source: sos.state.ia.us

---

**PREREQUISITE SKILL** Evaluate \(\frac{a}{1 - b}\) for the given values of \(a\) and \(b\). (Lesson 1-1)

80. \(a = 1, b = \frac{1}{2}\)
81. \(a = 3, b = -\frac{1}{2}\)
82. \(a = \frac{1}{3}, b = -\frac{1}{3}\)
83. \(a = \frac{1}{2}, b = \frac{1}{4}\)
84. \(a = -1, b = 0.5\)
85. \(a = 0.9, b = -0.5\)
Main Ideas
- Find the sum of an infinite geometric series.
- Write repeating decimals as fractions.

New Vocabulary
infinite geometric series
partial sum
convergent series

Infinite Geometric Series

Suppose you wrote a geometric series to find the sum of the heights of the rebounds of the ball on page 636. The series would have no last term because theoretically there is no last bounce of the ball. For every rebound of the ball, there is another rebound, 60% as high. Such a geometric series is called an **infinite geometric series**.

In the Bleachers  By Steve Moore

“And that, ladies and gentlemen, is the way the ball bounces.”

Infinite Geometric Series  Consider the infinite geometric series \(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\). You have already learned how to find the sum \(S_n\) of the first \(n\) terms of a geometric series. For an infinite series, \(S_n\) is called a **partial sum** of the series. The table and graph show some values of \(S_n\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(S_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(\frac{1}{2}) or 0.5</td>
</tr>
<tr>
<td>2</td>
<td>(\frac{3}{4}) or 0.75</td>
</tr>
<tr>
<td>3</td>
<td>(\frac{7}{8}) or 0.875</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{15}{16}) or 0.9375</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{31}{32}) or 0.96875</td>
</tr>
<tr>
<td>6</td>
<td>(\frac{63}{64}) or 0.984375</td>
</tr>
<tr>
<td>7</td>
<td>(\frac{127}{128}) or 0.9921875</td>
</tr>
</tbody>
</table>

Notice that as \(n\) increases, the partial sums level off and approach a limit of 1. This leveling-off behavior is characteristic of infinite geometric series for which \(|r| < 1\).

Let’s look at the formula for the sum of a finite geometric series and use it to find a formula for the sum of an infinite geometric series.
Formula for Sum if
\(-1 < r < 1\)
To convince yourself of this formula, make a table of the first ten partial sums of the geometric series with \(r = \frac{1}{2}\) and \(a_1 = 100\).

<table>
<thead>
<tr>
<th>Term Number</th>
<th>Term</th>
<th>Partial Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>100</td>
</tr>
<tr>
<td>2</td>
<td>50</td>
<td>150</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
<td>175</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>10</td>
<td>10</td>
<td>(\vdots)</td>
</tr>
</tbody>
</table>

Complete the table and compare the sum that the series is approaching to that obtained by using the formula.

**KEY CONCEPT**

*Sum of an Infinite Geometric Series*

The sum \(S\) of an infinite geometric series with \(-1 < r < 1\) is given by

\[ S = \frac{a_1}{1 - r}. \]

An infinite geometric series for which \(|r| \geq 1\) does not have a sum. Consider the series \(1 + 3 + 9 + 27 + 81 + \ldots\). In this series, \(a_1 = 1\) and \(r = 3\). The table shows some of the partial sums of this series. As \(n\) increases, \(S_n\) rapidly increases and has no limit.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(S_n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>121</td>
</tr>
<tr>
<td>10</td>
<td>29,524</td>
</tr>
<tr>
<td>15</td>
<td>7,174,453</td>
</tr>
<tr>
<td>20</td>
<td>1,743,392,200</td>
</tr>
</tbody>
</table>

**EXAMPLE**

*Sum of an Infinite Geometric Series*

Find the sum of each infinite geometric series, if it exists.

1. \(\frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \ldots\)

   **Step 1** Find the value of \(r\) to determine if the sum exists.
   \(a_1 = \frac{1}{2}\) and \(a_2 = \frac{3}{8}\), so \(r = \frac{3}{8} \cdot \frac{1}{2} = \frac{3}{4}\).
   Since \(|\frac{3}{4}| < 1\), the sum exists.

   **Step 2** Use the formula for the sum of an infinite geometric series.
   \[ S = \frac{a_1}{1 - r} \quad \text{Sum formula} \]
   \[ S = \frac{\frac{1}{2}}{1 - \frac{3}{4}} \quad a_1 = \frac{1}{2}, r = \frac{3}{4} \]
   \[ = \frac{\frac{1}{2}}{\frac{1}{4}} \quad \text{or} 2 \quad \text{Simplify.} \]

2. \(1 - 2 + 4 - 8 + \ldots\)

   \(a_1 = 1\) and \(a_2 = -2\), so \(r = \frac{-2}{1} = -2\). Since \(|-2| \geq 1\), the sum does not exist.

**CHECK Your Progress**

1A. \(3 + 9 + 27 + 51 + \ldots\)

1B. \(-3 + \frac{1}{3} - \frac{1}{27} + \ldots\)
You can use sigma notation to represent infinite series. An *infinity symbol* \( \infty \) is placed above the \( \Sigma \) to indicate that a series is infinite.

**EXAMPLE** Infinite Series in Sigma Notation

2. Evaluate \( \sum_{n=1}^{\infty} 24\left(-\frac{1}{5}\right)^{n-1} \).

\[
S = \frac{a_1}{1 - r} \quad \text{Sum formula}
\]
\[
= \frac{24}{1 - \left(-\frac{1}{5}\right)} \quad a_1 = 24, \ r = -\frac{1}{5}
\]
\[
= \frac{24}{6/5} \quad \text{Simplify.}
\]

**CHECK Your Progress**

2. Evaluate \( \sum_{n=1}^{\infty} 11\left(\frac{1}{3}\right)^{n-1} \).

Repeating Decimals The formula for the sum of an infinite geometric series can be used to write a repeating decimal as a fraction.

**EXAMPLE** Write a Repeating Decimal as a Fraction

Write 0.\( \overline{39} \) as a fraction.

Method 1

\[
0.\overline{39} = 0.393939\ldots
\]
\[
= 0.39 + 0.0039 + 0.000039 + \ldots
\]
\[
= \frac{39}{100} + \frac{39}{10,000} + \frac{39}{1,000,000} + \ldots
\]
\[
S = \frac{a_1}{1 - r} \quad \text{Sum formula}
\]
\[
= \frac{\frac{39}{100}}{1 - \frac{1}{100}} \quad a_1 = \frac{39}{100}, \ r = \frac{1}{100}
\]
\[
= \frac{\frac{39}{100}}{\frac{99}{100}} \quad \text{Subtract.}
\]
\[
= \frac{39}{99} \quad \text{or} \quad \frac{13}{33} \quad \text{Simplify.}
\]

Method 2

\[
S = 0.\overline{39} \quad \text{Label the given decimal.}
\]
\[
100S = 39.393939\ldots \quad \text{Repeating decimal}
\]
\[
99S = 39 \quad \text{Multiply each side by 100.}
\]
\[
99S = 39 \quad \text{Subtract the second equation from the third.}
\]
\[
S = \frac{39}{99} \quad \text{or} \quad \frac{13}{33} \quad \text{Divide each side by 99.}
\]

**CHECK Your Progress**

3. Write 0.47 as a fraction.
Infinite Geometric Series

Find the sum of each infinite geometric series, if it exists.

1. \(a_1 = 36, r = \frac{2}{3}\)
2. \(a_1 = 18, r = -1.5\)
3. \(16 + 24 + 36 + \cdots\)
4. \(\frac{1}{4} + \frac{1}{6} + \frac{1}{9} + \cdots\)

5. **CLOCKS** Altovese’s grandfather clock is broken. When she sets the pendulum in motion by holding it against the side of the clock and letting it go, it swings 24 centimeters to the other side, then 18 centimeters back, then 13.5 centimeters, and so on. What is the total distance that the pendulum swings before it stops?

Find the sum of each infinite geometric series, if it exists.

6. \(\sum_{n=1}^{\infty} 6 (-0.4)^{n-1}\)
7. \(\sum_{n=1}^{\infty} 40 \left(\frac{3}{5}\right)^{n-1}\)
8. \(\sum_{n=1}^{\infty} 35 \left(-\frac{3}{4}\right)^{n-1}\)
9. \(\sum_{n=1}^{\infty} \frac{1}{2} \left(\frac{3}{8}\right)^{n-1}\)

Write each repeating decimal as a fraction.

10. \(0.\overline{5}\)
11. \(0.\overline{73}\)
12. \(0.\overline{175}\)

For Exercises 32 and 33, refer to equilateral triangle \(ABC\), which has a perimeter of 39 centimeters. If the midpoints of the sides are connected, a smaller equilateral triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely.

32. Write an infinite geometric series to represent the sum of the perimeters of all of the triangles.
33. Find the sum of the perimeters of all of the triangles.

GEOMETRY
34. **PHYSICS** In a physics experiment, a steel ball on a flat track is accelerated and then allowed to roll freely. After the first minute, the ball has rolled 120 feet. Each minute the ball travels only 40% as far as it did during the preceding minute. How far does the ball travel?

35. Find the sum of each infinite geometric series, if it exists.
   - 34. $\frac{5}{3} - \frac{10}{9} + \frac{20}{27} - \ldots$
   - 35. $\frac{3}{2} - \frac{3}{4} + \frac{3}{8} - \ldots$
   - 36. $3 + 1.8 + 1.08 + \ldots$
   - 37. $1 - 0.5 + 0.25 - \ldots$
   - 38. $\sum_{n=1}^{\infty} 3(0.5)^n - 1$
   - 39. $\sum_{n=1}^{\infty} (1.5)(0.25)^n - 1$

36. Write each repeating decimal as a fraction
   - 40. 0.2\(\overline{46}\)
   - 41. 0.4\(\overline{27}\)
   - 42. 0.4\(\overline{27}\)
   - 43. 0.4\(\overline{5}\)
   - 44. 0.2\(\overline{31}\)

37. **SCIENCE MUSEUM** An exhibit at a science museum offers visitors the opportunity to experiment with the motion of an object on a spring. One visitor pulled the object down and let it go. The object traveled a distance of 1.2 feet upward before heading back the other way. Each time the object changed direction, it moved only 80% as far as it did in the previous direction. Find the total distance the object traveled.

46. The sum of an infinite geometric series is 81, and its common ratio is $\frac{2}{3}$. Find the first three terms of the series.

47. The sum of an infinite geometric series is 125, and the value of $r$ is 0.4. Find the first three terms of the series.

48. The common ratio of an infinite geometric series is $\frac{11}{16}$, and its sum is $76\frac{4}{5}$. Find the first four terms of the series.

49. The first term of an infinite geometric series is $-8$, and its sum is $-13\frac{1}{3}$. Find the first four terms of the series.

50. **OPEN ENDED** Write the series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ using sigma notation in two different ways.

51. **REASONING** Explain why $0.999999\ldots = 1$.

52. **FIND THE ERROR** Conrado and Beth are discussing the series $-\frac{1}{3} + \frac{4}{9} - \frac{16}{27} + \ldots$. Conrado says that the sum of the series is $-\frac{1}{7}$. Beth says that the series does not have a sum. Who is correct? Explain your reasoning.

53. **CHALLENGE** Derive the formula for the sum of an infinite geometric series by using the technique in Lessons 11-2 and 11-4. That is, write an equation for the sum $S$ of a general infinite geometric series, multiply each side of the equation by $r$, and subtract equations.

54. **Writing in Math** Use the information on page 650 to explain how an infinite geometric series applies to a bouncing ball. Explain how to find the total distance traveled, both up and down, by the bouncing ball described on page 636.
55. **ACT/SAT** What is the sum of an infinite geometric series with a first term of 6 and a common ratio of $\frac{1}{2}$?

A 3  
B 4  
C 9  
D 12

56. **REVIEW** What is the sum of the infinite geometric series

\[ \frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \ldots \]?

F $\frac{2}{3}$  
G 1  
H $1\frac{1}{3}$  
J $1\frac{2}{3}$

---

**Spiral Review**

Find $S_n$ for each geometric series described. (Lesson 11-4)

57. $a_1 = 1, a_6 = -243, r = -3$  
58. $a_1 = 72, r = \frac{1}{3}, n = 7$

59. **PHYSICS** A vacuum pump removes 20% of the air from a container with each stroke of its piston. What percent of the original air remains after five strokes? (Lesson 11-3)

Solve each equation or inequality. Check your solution. (Lesson 9-1)

60. $6^x = 216$  
61. $2^{2x} = \frac{1}{8}$  
62. $3^{x-2} \geq 27$

Simplify each expression. (Lesson 8-2)

63. $\frac{-2}{ab} + \frac{5}{a^2}$  
64. $\frac{1}{x-3} - \frac{2}{x+1}$  
65. $\frac{1}{x^2+6x+8} + \frac{3}{x+4}$

Write a quadratic equation with the given roots. Write the equation in the form $ax^2 + bx + c = 0$, where $a$, $b$, and $c$ are integers. (Lesson 5-3)

66. 6, −6  
67. −2, −7  
68. 6, 4

**RECREATION** For Exercises 69 and 70, refer to the graph at the right. (Lesson 2-3)

69. Find the average rate of change of the number of visitors to Yosemite National Park from 1998 to 2004.
70. Interpret your answer to Exercise 69.

**GET READY for the Next Lesson**

**PREREQUISITE SKILL** Find each function value. (Lesson 2-1)

71. $f(x) = 2x, f(1)$  
72. $g(x) = 3x - 3, g(2)$  
73. $h(x) = -2x + 2, h(0)$

74. $f(x) = 3x - 1, f\left(\frac{1}{2}\right)$  
75. $g(x) = x^2, g(2)$  
76. $h(x) = 2x^2 - 4, h(0)$
Find the indicated term of each arithmetic sequence. (Lesson 11-1)

1. \( a_1 = 7, \ d = 3, \ n = 14 \)
2. \( a_1 = 2, \ d = \frac{1}{2}, \ n = 8 \)

For Exercises 3 and 4, refer to the following information. (Lesson 11-1)

**READING** Amber makes a New Year’s resolution to read 50 books by the end of the year.

3. By the end of February, Amber has read 9 books. If she reads 3 books each month for the rest of the year, will she meet her goal? Explain.
4. If Amber has read 10 books by the end of April, how many will she have to read on average each month in order to meet her goal? (Lesson 11-2)

5. **MULTIPLE CHOICE** The figures below show a pattern of filled squares and white squares that can be described by a relationship between 2 variables.

![Figure 1](image1) ![Figure 2](image2) ![Figure 3](image3)

Which rule relates \( f \), the number of filled squares, to \( w \), the number of white squares? (Lesson 11-1)

- A \( w = f - 1 \)
- B \( w = 2f - 2 \)
- C \( f = \frac{1}{2}w - 1 \)
- D \( f = w - 1 \)

Find the sum of each arithmetic series described. (Lesson 11-2)

6. \( a_1 = 5, \ a_n = 29, \ n = 11 \)
7. \( 6 + 12 + 18 + \cdots + 96 \)

8. **BANKING** Veronica has a savings account with $1500 dollars in it. At the end of each month, the balance in her account has increased by 0.25%. How much money will Veronica have in her savings account at the end of one year? (Lesson 11-3)

9. **GAMES** In order to help members of a group get to know each other, sometimes the group plays a game. The first person states his or her name and an interesting fact about himself or herself. The next person must repeat the first person’s name and fact and then say his or her own. Each person must repeat the information for all those who preceded him or her. If there are 20 people in a group, what is the total number of times the names and facts will be stated? (Lesson 11-2)

10. Find \( a_7 \) for the geometric sequence 729, \(-243\), 81, \( \ldots \). (Lesson 11-3)

Find the sum of each geometric series, if it exists. (Lessons 11-4 and 11-5)

11. \( a_1 = 5, \ r = 3, \ n = 12 \)
12. \( 5 + 1 + \frac{1}{5} + \cdots \)
13. \( \sum_{n=1}^{6} 2(-3)^{n-1} \)
14. \( \sum_{n=1}^{\infty} 8\left(\frac{2}{3}\right)^{n-1} \)
15. \( \sum_{n=1}^{\infty} -13 \left(\frac{1}{3}\right)^{n-1} \)
16. \( \sum_{n=1}^{\infty} \frac{1}{100} \left(\frac{10}{9}\right)^{n-1} \)

Write each repeating decimal as a fraction. (Lesson 11-5)

17. 0.17 18. 0.256
19. 1.27 20. 3.1\overline{5}

**GEOMETRY** For Exercises 21 and 22, refer to square \( ABCD \), which has a perimeter of 120 inches. (Lesson 11-5)

If the midpoints of the sides are connected, a smaller square results. Suppose the process of connecting midpoints of sides and drawing new squares is continued indefinitely.

21. Write an infinite geometric series to represent the sum of the perimeters of all of the squares.

22. Find the sum of the perimeters of all of the squares.
Spreadsheet Lab
Amortizing Loans

When a payment is made on a loan, part of the payment is used to cover the interest that has accumulated since the last payment. The rest is used to reduce the principal, or original amount of the loan. This process is called amortization. You can use a spreadsheet to analyze the payments, interest, and balance on a loan.

EXAMPLE

Marisela just bought a new sofa for $495. The store is letting her make monthly payments of $43.29 at an interest rate of 9% for one year. How much will she still owe after six months?

Every month, the interest on the remaining balance will be \( \frac{9\%}{12} \) or 0.75%.

You can find the balance after a payment by multiplying the balance after the previous payment by \( 1 + 0.0075 \) or 1.0075 and then subtracting 43.29.

In a spreadsheet, use the column of numbers for the number of payments and use column B for the balance. Enter the interest rate and monthly payment in cells in column A so that they can be easily updated if the information changes.

The spreadsheet shows the formulas for the balances after each of the first six payments. After six months, Marisela still owes $253.04.

EXERCISES

1. Let \( b_n \) be the balance left on Marisela’s loan after \( n \) months. Write an equation relating \( b_n \) and \( b_{n+1} \).
2. What percent of Marisela’s loan remains to be paid after half a year?
3. Extend the spreadsheet to the whole year. What is the balance after 12 payments? Why is it not 0?
4. Suppose Marisela decides to pay $50 every month. How long would it take her to pay off the loan?
5. Suppose that, based on how much she can afford, Marisela will pay a variable amount each month in addition to the $43.29. Explain how the flexibility of a spreadsheet can be used to adapt to this situation.
6. Jamie has a three-year, $12,000 car loan. The annual interest rate is 6%, and his monthly payment is $365.06. After twelve months, he receives an inheritance which he wants to use to pay off the loan. How much does he owe at that point?
A shoot on a sneezewort plant must grow for two months before it is strong enough to put out another shoot. After that, it puts out at least one shoot every month.

Special Sequences Notice that each term in the sequence is the sum of the two previous terms. For example, \(8 = 3 + 5\) and \(13 = 5 + 8\). This sequence is called the Fibonacci sequence, and it is found in many places in nature.

The formula \(a_n = a_{n-2} + a_{n-1}\) is an example of a recursive formula. This means that each term is formulated from one or more previous terms.

**EXAMPLE** Use a Recursive Formula

Find the first five terms of the sequence in which \(a_1 = 4\) and \(a_{n+1} = 3a_n - 2, n \geq 1\).

\[
\begin{align*}
a_1 &= 4 \\
a_{n+1} &= 3a_n - 2 \quad \text{Recursive formula} \\
a_2 &= 3(4) - 2 \quad n = 1 \\
a_2 &= 10 \\
a_3 &= 3a_2 - 2 \quad n = 2 \\
a_3 &= 3(10) - 2 \quad a_2 = 10 \\
a_3 &= 28 \\
a_4 &= 3a_3 - 2 \quad n = 3 \\
a_4 &= 3(28) - 2 \quad a_3 = 28 \\
a_4 &= 82 \\
a_5 &= 3a_4 - 2 \quad n = 4 \\
a_5 &= 3(82) - 2 \quad a_4 = 82 \\
a_5 &= 244 \\
\end{align*}
\]

The first five terms of the sequence are 4, 10, 28, 82, and 244.

1. Find the first five terms of the sequence in which \(a_1 = -1\) and \(a_{n+1} = 2a_n + 4, n \geq 1\).
MEDICAL RESEARCH A pharmaceutical company is experimenting with a new drug. An experiment begins with $1.0 \times 10^9$ bacteria. A dose of the drug that is administered every four hours can kill $4.0 \times 10^8$ bacteria. Between doses of the drug, the number of bacteria increases by 50%.

a. Write a recursive formula for the number of bacteria alive before each application of the drug.

Let $b_n$ represent the number of bacteria alive just before the $n$th application of the drug. $4.0 \times 10^8$ of these will be killed by the drug, leaving $b_n - 4.0 \times 10^8$. The number $b_{n+1}$ of bacteria before the next application will have increased by 50%. So $b_{n+1} = 1.5(b_n - 4.0 \times 10^8)$, or $1.5b_n - 6.0 \times 10^8$.

b. Find the number of bacteria alive before the fifth application.

Before the first application of the drug, there were $1.0 \times 10^9$ bacteria alive, so $b_1 = 1.0 \times 10^9$.

For $n = 1$, $b_{1+1} = 1.5b_1 - 6.0 \times 10^8$

For $n = 2$, $b_{2+1} = 1.5b_2 - 6.0 \times 10^8$

For $n = 3$, $b_{3+1} = 1.5b_3 - 6.0 \times 10^8$

For $n = 4$, $b_{4+1} = 1.5b_4 - 6.0 \times 10^8$

For $n = 5$, $b_{5+1} = 1.5b_5 - 6.0 \times 10^8$

Before the fifth dose, there would be $1.875 \times 10^8$ bacteria alive.

A stronger dose of the drug can kill $6.0 \times 10^8$ bacteria.

2A. Write a recursive formula for the number of bacteria alive before each dose of the drug.

2B. How many of the stronger doses of the drug will kill all the bacteria?

Real-World Link In 1928, Alexander Fleming found that penicillin mold could destroy certain types of bacteria. Production increases allowed the price of penicillin to fall from about $20 per dose in 1943 to $0.55 per dose in 1946.

Source: inventors.about.com

Special Sequences

The object of the Towers of Hanoi game is to move a stack of $n$ coins from one position to another in the fewest number $a_n$ of moves with these rules.

- You may only move one coin at a time.
- A coin must be placed on top of another coin, not underneath.
- A smaller coin may be placed on top of a larger coin, but not vice versa. For example, a penny may not be placed on top of a dime.

(continued on the next page)
Look Back
To review the composition of functions, see Lesson 7-5.

**MODEL AND ANALYZE**

1. Draw three circles on a sheet of paper, as shown. Place a penny on the first circle. What is the least number of moves required to get the penny to the second circle?

2. Place a nickel and a penny on the first circle, with the penny on top. What is the least number of moves that you can make to get the stack to another circle? (Remember, a nickel cannot be placed on top of a penny.)

3. Place a nickel, penny, and dime on the first circle. What is the least number of moves that you can take to get the stack to another circle?

**MAKE A CONJECTURE**

4. Place a quarter, nickel, penny, and dime on the first circle. Experiment to find the least number of moves needed to get the stack to another circle. Make a conjecture about a formula for the minimum number \( a_n \) of moves required to move a stack of \( n \) different sized coins.

---

**Iteration**

Iteration is the process of composing a function with itself repeatedly. For example, if you compose a function with itself once, the result is \( f \circ f(x) \) or \( f(f(x)) \). If you compose a function with itself two times, the result is \( f \circ f \circ f(x) \) or \( f(f(f(x))) \), and so on.

You can use iteration to recursively generate a sequence. Start with an initial value \( x_0 \). Let \( x_1 = f(x_0) \), \( x_2 = f(x_1) \) or \( f(f(x_0)) \), \( x_3 = f(x_2) \) or \( f(f(f(x_0))) \), and so on.

**EXAMPLE**

**Iterate a Function**

Find the first three iterates \( x_1, x_2, \) and \( x_3 \) of the function \( f(x) = 2x + 3 \) for an initial value of \( x_0 = 1 \).

\[
\begin{align*}
x_1 &= f(x_0) \\
&= f(1) \\
&= 2(1) + 3 \text{ or } 5 \\
&\text{Iterate the function.} \\
x_2 &= f(x_1) \\
&= f(5) \\
&= 2(5) + 3 \text{ or } 13 \\
&\text{Iterate the function.} \\
x_3 &= f(x_2) \\
&= f(13) \\
&= 2(13) + 3 \text{ or } 29 \\
&\text{Iterate the function.}
\end{align*}
\]

The first three iterates are 5, 13, and 29.

**CHECK Your Progress**

3. Find the first four iterates, \( x_1, x_2, x_3, x_4 \), of the function \( f(x) = x^2 - 2x - 1 \) for an initial value of \( x_0 = -1 \).

---

**Example 1**

Find the first five terms of each sequence.

1. \( a_1 = 12, a_{n+1} = a_n - 3 \)
2. \( a_1 = -3, a_{n+1} = a_n + n \)
3. \( a_1 = 0, a_{n+1} = -2a_n - 4 \)
4. \( a_1 = 1, a_2 = 2, a_{n+2} = 4a_{n+1} - 3a_n \)
Find the first five terms of each sequence.

10. \( a_1 = -6, \ a_{n+1} = a_n + 3 \)

11. \( a_1 = 13, \ a_{n+1} = a_n + 5 \)

12. \( a_1 = 2, \ a_{n+1} = a_n - n \)

13. \( a_1 = 6, \ a_{n+1} = a_n + n + 3 \)

14. \( a_1 = 9, \ a_{n+1} = 2a_n - 4 \)

15. \( a_1 = 4, \ a_{n+1} = 3a_n - 6 \)

16. If \( a_0 = 7 \) and \( a_{n+1} = a_n + 12 \) for \( n \geq 0 \), find the value of \( a_5 \).

17. If \( a_0 = 1 \) and \( a_{n+1} = -2.1 \) for \( n \geq 0 \), then what is the value of \( a_4 \)?

Find the first three iterates of each function for the given initial value.

7. \( f(x) = 3x - 4, \ x_0 = 3 \)

8. \( f(x) = -2x + 5, \ x_0 = 2 \)

9. \( f(x) = x^2 + 2, \ x_0 = -1 \)

Find the first five terms of each sequence.

18. \( f(x) = 9x - 2, \ x_0 = 2 \)

19. \( f(x) = 4x - 3, \ x_0 = 2 \)

20. \( f(x) = 3x + 5, \ x_0 = -4 \)

21. \( f(x) = 5x + 1, \ x_0 = -1 \)

GEOMETRY For Exercises 22–24, use the following information.

Join two 1-unit by 1-unit squares to form a rectangle. Next, draw a larger square along a long side of the rectangle. Continue this process.

22. Write the sequence of the lengths of the sides of the squares you added at each step. Begin the sequence with two original squares.

23. Write a recursive formula for the sequence of lengths added.

24. Identify the sequence in Exercise 23.

GEOMETRY For Exercises 25–27, study the triangular numbers shown below.

25. Write a sequence of the first five triangular numbers.

26. Write a recursive formula for the \( n \)th triangular number \( t_n \).

27. What is the 200th triangular number?

28. LOANS Miguel’s monthly car payment is $234.85. The recursive formula \( b_n = 1.005b_{n-1} - 234.85 \) describes the balance left on the loan after \( n \) payments. Find the balance of the $10,000 loan after each of the first eight payments.
29. **ECONOMICS** If the rate of inflation is 2%, the cost of an item in future years can be found by iterating the function \( c(x) = 1.02x \). Find the cost of a $70 MP3 player in four years if the rate of inflation remains constant.

Find the first three iterates of each function for the given initial value.

30. \( f(x) = 2x^2 - 5 \), \( x_0 = -1 \)  
31. \( f(x) = 3x^2 - 4 \), \( x_0 = 1 \)  
32. \( f(x) = 2x^2 + 2x + 1 \), \( x_0 = \frac{1}{2} \)  
33. \( f(x) = 3x^2 - 3x + 2 \), \( x_0 = \frac{1}{3} \)

34. **OPEN ENDED** Write a recursive formula for a sequence whose first three terms are 1, 1, and 3.

35. **REASONING** Is the statement \( x_n \neq x_{n-1} \) *always*, *sometimes*, or *never* true if \( x_n = f(x_{n-1}) \)? Explain.

36. **CHALLENGE** Are there a function \( f(x) \) and an initial value \( x_0 \) such that the first three iterates, in order, are 4, 4, and 7? Explain.

37. **Writing in Math** Use the information on page 658 to explain how the Fibonacci sequence is illustrated in nature. Include the 13th term in the sequence, with an explanation of what it tells you about the plant described.

38. **ACT/SAT** The figure is made of three concentric semicircles. What is the total area of the shaded regions?

\[
\begin{align*}
A & : 4\pi \text{ units}^2 & C & : 12\pi \text{ units}^2 \\
B & : 10\pi \text{ units}^2 & D & : 20\pi \text{ units}^2
\end{align*}
\]

39. **REVIEW** If \( x \) is a real number, for what values of \( x \) is the equation \( \frac{4x - 16}{4} = x - 4 \) true?

\[
F \quad \text{all values of } x \\
G \quad \text{some values of } x \\
H \quad \text{no values of } x \\
J \quad \text{impossible to determine}
\]

---

**Spiral Review**

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

40. \( 9 + 6 + 4 + \cdots \)  
41. \( \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots \)  
42. \( 4 - \frac{8}{3} + \frac{16}{9} + \cdots \)

Find the sum of each geometric series. (Lesson 11-4)

43. \( 2 - 10 + 50 - \cdots \) to 6 terms  
44. \( 3 + \frac{1}{3} + \cdots \) to 7 terms

45. **GEOMETRY** The area of rectangle \( ABCD \) is \( 6x^2 + 38x + 56 \) square units. Its width is \( 2x + 8 \) units. What is the length of the rectangle? (Lesson 6-3)

46. \( 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \)  
47. \( \frac{4 \cdot 3}{2 \cdot 1} \)  
48. \( \frac{9 \cdot 8 \cdot 7 \cdot 6}{4 \cdot 3 \cdot 2 \cdot 1} \)

---

**EXTRA PRACTICE** 
See pages 915, 936.

**Mathonline** 
Self-Check Quiz at algebra2.com

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**PRACTICE**
Fractals are sets of points that often involve intricate geometric shapes. Many fractals have the property that when small parts are magnified, the detail of the fractal is not lost. In other words, the magnified part is made up of smaller copies of itself. Such fractals can be constructed recursively.

You can use isometric dot paper to draw stages of the construction of a fractal called the von Koch snowflake.

**ACTIVITY**

**Stage 1** Draw an equilateral triangle with sides of length 9 units on the dot paper.

**Stage 2** Now remove the middle third of each side of the triangle from Stage 1 and draw the other two sides of an equilateral triangle pointing outward.

Imagine continuing this process infinitely. The von Koch snowflake is the shape that these stages approach.

**Model and Analyze the Results**

1. Copy and complete the table. Draw stage 3, if necessary.

<table>
<thead>
<tr>
<th>Stage</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Segments</td>
<td>3</td>
<td>8</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>Length of each Segment</td>
<td>9</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>
| Perimeter | 27 | 36 | ... | ...

2. Write recursive formulas for the number $s_n$ of segments in Stage $n$, the length $\ell_n$ of each segment in Stage $n$, and the perimeter $P_n$ of Stage $n$.

3. Write nonrecursive formulas for $s_n$, $\ell_n$, and $P_n$.

4. What is the perimeter of the von Koch snowflake? Explain.

5. Explain why the area of the von Koch snowflake can be represented by the infinite series $\frac{81\sqrt{3}}{4} + \frac{27\sqrt{3}}{4} + 3\sqrt{3} + \frac{4\sqrt{3}}{3} + \ldots$.

6. Find the sum of the series in Exercise 5. Explain your steps.

7. Do you think the results of Exercises 4 and 6 are contradictory? Explain.
According to the U.S. Census Bureau, ten percent of families have three or more children. If a family has four children, there are six sequences of births of boys and girls that result in two boys and two girls. These sequences are listed below.

BBGG  BGBG  BGGB  GBBG  GBGB  GGBB

**Pascal’s Triangle** You can use the coefficients in powers of binomials to count the number of possible sequences in situations such as the one above. Expand a few powers of the binomial \( b + g \).

\[
(b + g)^0 = 1
\]
\[
(b + g)^1 = 1b^1g^0 + 1b^0g^1
\]
\[
(b + g)^2 = 1b^2g^0 + 2b^1g^1 + 1b^0g^2
\]
\[
(b + g)^3 = 1b^3g^0 + 3b^2g^1 + 3b^1g^2 + 1b^0g^3
\]
\[
(b + g)^4 = 1b^4g^0 + 4b^3g^1 + 6b^2g^2 + 4b^1g^3 + 1b^0g^4
\]

The coefficient 4 of the \( b^1g^3 \) term in the expansion of \((b + g)^4\) gives the number of sequences of births that result in one boy and three girls.

Here are some patterns in any binomial expansion of the form \((a + b)^n\).

1. There are \( n + 1 \) terms.
2. The exponent \( n \) of \((a + b)^n\) is the exponent of \( a \) in the first term and the exponent of \( b \) in the last term.
3. In successive terms, the exponent of \( a \) decreases by one, and the exponent of \( b \) increases by one.
4. The sum of the exponents in each term is \( n \).
5. The coefficients are symmetric. They increase at the beginning of the expansion and decrease at the end.

The coefficients form a pattern that is often displayed in a triangular formation. This is known as **Pascal’s triangle**. Notice that each row begins and ends with 1. Each coefficient is the sum of the two coefficients above it in the previous row.
The Binomial Theorem

Another way to show the coefficients in a binomial expansion is to write them in terms of the previous coefficients.

This pattern is summarized in the Binomial Theorem.

**Binomial Theorem**

If \( n \) is a nonnegative integer, then

\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k
\]

**EXAMPLE**

**Use the Binomial Theorem**

Expand \((a - b)^6\).

Use the sequence \(1, \frac{6}{1}, \frac{6 \cdot 5}{1 \cdot 2}, \frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \) to find the coefficients for the first four terms. Then use symmetry to find the remaining coefficients.

\[
(a - b)^6 = a^6 - 6a^5 b + 15a^4 b^2 - 20a^3 b^3 + 15a^2 b^4 - 6ab^5 + b^6
\]

**CHECK Your Progress**

2. Expand \((w + z)^5\).
The factors in the coefficients of binomial expansions involve special products called **factorials**. For example, the product $4 \cdot 3 \cdot 2 \cdot 1$ is written $4!$ and is read $4$ factorial. In general, if $n$ is a positive integer, then $n! = n(n-1)(n-2)(n-3) \ldots 2 \cdot 1$.  
*By definition, $0! = 1$.]*

**EXAMPLE**  
**Factorials**  

Evaluate $\frac{8!}{3!5!}$.  

\[
\frac{8!}{3!5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{3 \cdot 2 \cdot 1 \cdot 5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2} = 56\]  

**Check Your Progress**  
3. Evaluate $\frac{12!}{8!4!}$.  

The Binomial Theorem can be written in factorial notation and in sigma notation.  

**KEY CONCEPT**  
**Binomial Theorem, Factorial Form**  
\[
(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \]  

**EXAMPLE**  
**Use a Factorial Form of the Binomial Theorem**  

Expand $(2x + y)^5$.  

\[
(2x + y)^5 = \sum_{k=0}^{5} \binom{5}{k} (2x)^{5-k} y^k \]  

\[
= \binom{5}{0} (2x)^5 y^0 + \binom{5}{1} (2x)^4 y + \binom{5}{2} (2x)^3 y^2 + \binom{5}{3} (2x)^2 y^3 + \binom{5}{4} (2x)^1 y^4 + \binom{5}{5} y^5 \]  

Let $k = 0, 1, 2, 3, 4,$ and $5$.  

\[
= \frac{5!}{0!5!} (2x)^5 y^0 + \frac{5!}{5!0!} (2x)^4 y + \frac{5!}{4!1!} (2x)^3 y^2 + \frac{5!}{3!2!} (2x)^2 y^3 + \frac{5!}{2!3!} (2x)^1 y^4 + \frac{5!}{1!4!} (2x)^0 y^5 \]  

\[
= \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} (2x)^5 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 3 \cdot 2 \cdot 1} (2x)^4 y + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{3 \cdot 2 \cdot 1 \cdot 2 \cdot 1} (2x)^3 y^2 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 3 \cdot 2 \cdot 1} (2x)^2 y^3 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 4 \cdot 3 \cdot 2 \cdot 1} (2x) y^4 + \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{1 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} y^5 \]  

\[
= 32x^5 + 80x^4 y + 80x^3 y^2 + 40x^2 y^3 + 10xy^4 + y^5 \]  

Simplify.  

**Check Your Progress**  
4. Expand $(q - 3r)^4$.  

---

**Study Tip**  
**Graphing Calculators**  
On a TI-83/84 Plus, the factorial symbol, $!$, is located on the MATH PRB menu.
Sometimes you need to know only a particular term of a binomial expansion. Note that when the Binomial Theorem is written in sigma notation, $k = 0$ for the first term, $k = 1$ for the second term, and so on. In general, the value of $k$ is always one less than the number of the term you are finding.

**EXAMPLE**

**Find a Particular Term**

Find the fifth term in the expansion of $(p + q)^{10}$.

First, use the Binomial Theorem to write the expansion in sigma notation.

$$(p + q)^{10} = \sum_{k=0}^{10} \frac{10!}{(10-k)!k!} p^{10-k} q^k$$

In the fifth term, $k = 4$.

$$\frac{10!}{(10-4)!4!} p^{10-4} q^4 = \frac{10!}{6!4!} p^6 q^4$$

$$= 210 p^6 q^4$$

Simplify.

5. Find the eighth term in the expansion of $(x - y)^{12}$.

**CHECK Your Progress**

Expand each power.

1. $(p + q)^5$
2. $(t + 2)^6$
3. $(x - 3y)^4$

4. **GEOMETRY** Write an expanded expression for the volume of the cube at the right.

Evaluate each expression.

5. $8!$
6. $10!$
7. $\frac{13!}{9!}$
8. $\frac{12!}{2!10!}$

Find the indicated term of each expansion.

9. fourth term of $(a + b)^8$
10. fifth term of $(2a + 3b)^{10}$

**Exercises**

Expand each power.

11. $(a - b)^3$
12. $(m + n)^4$
13. $(r + s)^8$
14. $(m - a)^5$
15. $(x + 3)^5$
16. $(a - 2)^4$

Evaluate each expression.

17. $9!$
18. $13!$
19. $\frac{9!}{7!}$
20. $\frac{7!}{4!}$
Find the indicated term of each expansion.

21. sixth term of \((x - y)^9\)  
22. seventh term of \((x + y)^{12}\)
23. fourth term of \((x + 2)^7\)  
24. fifth term of \((a - 3)^8\)

25. **SCHOOL**  Mr. Hopkins is giving a five-question true-false quiz. How many ways could a student answer the questions with three trues and two falses?

26. **INTRAMURALS**  Ofelia is taking ten shots in the intramural free-throw shooting competition. How many sequences of makes and misses are there that result in her making eight shots and missing two?

Expand each power.

27. \((2b - x)^4\)  
28. \((2a + b)^6\)  
29. \((3x - 2y)^5\)  
30. \((3x + 2y)^4\)  
31. \(\left(\frac{a}{2} + 2\right)^5\)  
32. \(\left(3 + \frac{m}{3}\right)^5\)

Evaluate each expression.

33. \(\frac{12!}{8!4!}\)  
34. \(\frac{14!}{5!9!}\)

Find the indicated term of each expansion.

35. fifth term of \((2a + 3b)^{10}\)  
36. fourth term of \((2x + 3y)^9\)
37. fourth term of \(\left(x + \frac{1}{3}\right)^7\)  
38. sixth term of \(\left(x - \frac{1}{2}\right)^{10}\)

39. **GENETICS**  The color of a particular flower may be either red, white, or pink. If the flower has two red alleles \(R\), the flower is red. If the flower has two white alleles \(w\), the flower is white. If the flower has one allele of each color, the flower will be pink. In a lab, two pink flowers are mated and eventually produce 1000 offspring. How many of the 1000 offspring will be pink?

40. **GAMES**  The diagram shows the board for a game in which disks are dropped down a chute. A pattern of nails and dividers causes the disks to take various paths to the sections at the bottom. How many paths through the board lead to each bottom section?

41. **OPEN ENDED**  Write a power of a binomial for which the first term of the expansion is \(625x^4\).

42. **CHALLENGE**  Explain why \(\frac{12!}{7!5!} + \frac{12!}{6!6!} = \frac{13!}{7!6!}\) without finding the value of any of the expressions.

43. **Writing in Math**  Use the information on page 664 to explain how the power of a binomial describes the number of boys and girls in a family. Include the expansion of \((b + g)^5\) and an explanation of what it tells you about sequences of births of boys and girls in families with five children.
44. ACT/SAT If four lines intersect as shown, what is the value of $x + y$?

![Diagram of four lines intersecting at 75°, 145°, 70°, and 115° angles.]

45. REVIEW $(2x - 2)^4 =$

- F $16x^4 + 64x^3 - 96x^2 - 64x + 16$
- G $16x^4 - 32x^3 - 192x^2 - 64x + 16$
- H $16x^4 - 64x^3 + 96x^2 - 64x + 16$
- J $16x^4 + 32x^3 - 192x^2 - 64x + 16$

A  70  B  115  C  140  D  220

Spiral Review

Find the first five terms of each sequence. (Lesson 11-6)

46. $a_1 = 7, a_{n+1} = a_n - 2$

47. $a_1 = 3, a_{n+1} = 2a_n - 1$

48. MINIATURE GOLF A wooden pole swings back and forth over the cup on a miniature golf hole. One player pulls the pole to the side and lets it go. Then it follows a swing pattern of 25 centimeters, 20 centimeters, 16 centimeters, and so on until it comes to rest. What is the total distance the pole swings before coming to rest? (Lesson 11-5)

Without writing the equation in standard form, state whether the graph of each equation is a parabola, circle, ellipse, or hyperbola. (Lesson 10-6)

49. $x^2 - 6x - y^2 - 3 = 0$

50. $4y - x + y^2 = 1$

Express each logarithm in terms of common logarithms. Then approximate its value to four decimal places. (Lesson 9-4)

51. $\log_2 5$

52. $\log_3 10$

53. $\log_5 8$

Determine any vertical asymptotes and holes in the graph of each rational function. (Lesson 8-3)

54. $f(x) = \frac{1}{x^2 + 5x + 6}$

55. $f(x) = \frac{x + 2}{x^2 + 3x - 4}$

56. $f(x) = \frac{x^2 + 4x + 3}{x + 3}$

GET READY for the Next Lesson

PREREQUISITE SKILL State whether each statement is true or false when $n = 1$. Explain. (Lesson 1-1)

57. $1 = \frac{n(n + 1)}{2}$
58. $1 = \frac{(n + 1)(2n + 1)}{2}$
59. $1 = \frac{n^2 (n + 1)^2}{4}$

60. $3^n - 1$ is even.
61. $7^n - 3^n$ is divisible by 4.
62. $2^n - 1$ is prime.
Imagine the positive integers as a ladder that goes upward forever. You know that you cannot leap to the top of the ladder, but you can stand on the first step, and no matter which step you are on, you can always climb one step higher. Is there any step you cannot reach?

**Mathematical Induction**

Mathematical induction is used to prove statements about positive integers. This proof uses three steps.

**Key Concept**

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Show that the statement is true for some positive integer ( n ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step 2</td>
<td>Assume that the statement is true for some positive integer ( k ), where ( k \geq n ). This assumption is called the inductive hypothesis.</td>
</tr>
<tr>
<td>Step 3</td>
<td>Show that the statement is true for the next positive integer ( k + 1 ). If so, we can assume that the statement is true for any positive integer.</td>
</tr>
</tbody>
</table>

---

**Example**

**Summation Formula**

Prove that \( \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \).

**Step 1**

When \( n = 1 \), the left side of the given equation is \( 1^2 \) or 1.

The right side is \( \frac{1(1+1)(2(1)+1)}{6} \) or 1. Thus, the equation is true for \( n = 1 \).

**Step 2**

Assume \( \sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6} \) for a positive integer \( k \).

**Step 3**

Show that the given equation is true for \( n = k + 1 \).

\[
\begin{align*}
1^2 + 2^2 + 3^2 + \ldots + k^2 + (k + 1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k + 1)^2 \\
&= \frac{k(k+1)(2k+1) + 6(k+1)^2}{6} \\
&= \frac{(k + 1)[k(2k+1) + 6(k+1)]}{6} \\
&= \frac{(k + 1)[2k^2 + 7k + 6]}{6} \\
&= \frac{(k + 1)(k + 2)(2k + 3)}{6} \\
&= \frac{(k + 1)(k + 1 + 1)(2k + 1)}{6}
\end{align*}
\]

Add \( (k + 1)^2 \) to each side.
Add.
Factor.
Simplify.
Factor.

---

**Main Ideas**

- Prove statements by using mathematical induction.
- Disprove statements by finding a counterexample.

**New Vocabulary**

- mathematical induction
- inductive hypothesis
The last expression is the right side of the equation to be proved, where \( n \) has been replaced by \( k + 1 \). Thus, the equation is true for \( n = k + 1 \). This proves the conjecture.

Example: Divisibility

Prove that \( 7^n - 1 \) is divisible by 6 for all positive integers \( n \).

Step 1 When \( n = 1 \), \( 7^n - 1 = 7^1 - 1 \) or 6. Since 6 is divisible by 6, the statement is true for \( n = 1 \).

Step 2 Assume that \( 7^k - 1 \) is divisible by 6 for some positive integer \( k \). This means that there is a whole number \( r \) such that \( 7^k - 1 = 6r \).

Step 3 Show that the statement is true for \( n = k + 1 \).

\[
\begin{align*}
7^k - 1 &= 6r \\
7^k &= 6r + 1 \\
7(7^k) &= 7(6r + 1) \\
7^{k+1} &= 42r + 7 \\
7^{k+1} - 1 &= 42r + 6 \\
7^{k+1} - 1 &= 6(7r + 1)
\end{align*}
\]

Since \( r \) is a whole number, \( 7r + 1 \) is a whole number. Therefore, \( 7^{k+1} - 1 \) is divisible by 6. Thus, the statement is true for \( n = k + 1 \).

This proves that \( 7^n - 1 \) is divisible by 6 for all positive integers \( n \).

Example: Counterexample

Find a counterexample for \( 1^4 + 2^4 + 3^4 + \cdots + n^4 = 1 + (4n - 4)^2 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>Left Side of Formula</th>
<th>Right Side of Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( 1^4 ) or 1</td>
<td>( 1 + [4(1) - 4]^2 = 1 + 0^2 ) or 1 true</td>
</tr>
<tr>
<td>2</td>
<td>( 1^4 + 2^4 = 1 + 16 ) or 17</td>
<td>( 1 + [4(2) - 4]^2 = 1 + 4^2 ) or 17 true</td>
</tr>
<tr>
<td>3</td>
<td>( 1^4 + 2^4 + 3^4 = 1 + 16 + 81 ) or 98</td>
<td>( 1 + [4(3) - 4]^2 = 1 + 64 ) or 65 false</td>
</tr>
</tbody>
</table>

The value \( n = 3 \) is a counterexample for the equation.

Example: Counterexample

Find a counterexample for the statement that \( 2n^2 + 11 \) is prime for all positive integers \( n \).
Prove that each statement is true for all positive integers.

1. \(1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}\)

2. \(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}\)

3. **PARTIES** Suppose that each time a new guest arrives at a party, he or she shakes hands with each person already at the party. Prove that after \(n\) guests have arrived, a total of \(\frac{n(n - 1)}{2}\) handshakes have taken place.

Prove that each statement is true for all positive integers.

4. \(4^n - 1\) is divisible by 3.

5. \(5^n + 3\) is divisible by 4.

Find a counterexample for each statement.

6. \(1 + 2 + 3 + \cdots + n = n^2\)

7. \(2^n + 3^n\) is divisible by 4.

8. \(1 + 5 + 9 + \cdots + (4n - 3) = n(2n - 1)\)

9. \(2 + 5 + 8 + \cdots + (3n - 1) = \frac{n(3n + 1)}{2}\)

10. \(1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}\)

11. \(1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{n(2n - 1)(2n + 1)}{3}\)

12. \(8^n - 1\) is divisible by 7.

13. \(9^n - 1\) is divisible by 8.

14. **ARCHITECTURE** A memorial being constructed in a city park will be a brick wall, with a top row of six gold-plated bricks engraved with the names of six local war veterans. Each row has two more bricks than the row above it. Prove that the number of bricks in the top \(n\) rows is \(n^2 + 5n\).

15. **NATURE** The terms of the Fibonacci sequence are found in many places in nature. The number of spirals of seeds in sunflowers are Fibonacci numbers, as are the number of spirals of scales on a pinecone. The Fibonacci sequence begins 1, 1, 2, 3, 5, 8, ... Each element after the first two is found by adding the previous two terms. If \(f_n\) stands for the \(n\)th Fibonacci number, prove that \(f_1 + f_2 + \cdots + f_n = f_{n + 2} - 1\).

Find a counterexample for each statement.

16. \(1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(3n - 1)}{2}\)

17. \(1^3 + 3^3 + 5^3 + \cdots + (2n - 1)^3 = 12n^3 - 23n^2 + 12n\)

18. \(3^n + 1\) is divisible by 4.

19. \(2^n + 2n^2\) is divisible by 4.

20. \(n^2 - n + 11\) is prime.

21. \(n^2 + n + 41\) is prime.
Prove that each statement is true for all positive integers.

22. \[ \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^n} = \frac{1}{2} \left(1 - \frac{1}{3^n}\right) \]

23. \[ \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots + \frac{1}{4^n} = \frac{1}{3} \left(1 - \frac{1}{4^n}\right) \]

24. \[ 12^n + 10 \text{ is divisible by } 11. \]

25. \[ 13^n + 11 \text{ is divisible by } 12. \]

26. **ARITHMETIC SERIES** Use mathematical induction to prove the formula 
   \[ a_1 + (a_1 + d) + (a_1 + 2d) + \cdots + [a_1 + (n-1)d] = \frac{n}{2} [2a_1 + (n-1)d] \]
   for the sum of an arithmetic series.

27. **GEOMETRIC SERIES** Use mathematical induction to prove the formula 
   \[ a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-1} = \frac{a_1(1 - r^n)}{1 - r} \]
   for the sum of a finite geometric series.

28. **PUZZLES** Show that a \(2^n\) by \(2^n\) checkerboard with the top right square missing can always be covered by nonoverlapping L-shaped tiles like the one at the right.

29. **OPEN ENDED** Write an expression of the form \(b^n - 1\) that is divisible by 2 for all positive integers \(n\).

30. **CHALLENGE** Refer to Example 2. Explain how to use the Binomial Theorem to show that \(7^n - 1\) is divisible by 6 for all positive integers \(n\).

31. **Writing in Math** Use the information on page 670 to explain how the concept of a ladder can help you prove statements about numbers.

---

32. **ACT/SAT** PQRS is a square. What is the ratio of the length of diagonal QS to the length of side RS?

   A. 2
   B. \(\sqrt{2}\)
   C. 1
   D. \(\frac{\sqrt{2}}{2}\)

33. **REVIEW** The lengths of the bases of an isosceles trapezoid are 15 centimeters and 29 centimeters. If the perimeter of this trapezoid is 94 centimeters, what is the area?

   F. 500 cm\(^2\)
   H. 528 cm\(^2\)
   G. 515 cm\(^2\)
   J. 550 cm\(^2\)

---

34. Expand each power. (Lesson 11-7)

   34. \((x + y)^6\)
   35. \((a - b)^7\)
   36. \((2x + y)^8\)

37. Find the first three iterates of each function for the given initial value. (Lesson 11-6)

   37. \(f(x) = 3x - 2, x_0 = 2\)
   38. \(f(x) = 4x^2 - 2, x_0 = 1\)

39. **BIOLOGY** Suppose an amoeba divides into two amoebas once every hour. How long would it take for a single amoeba to become a colony of 4096 amoebas? (Lesson 9-2)
Key Vocabulary

- arithmetic means (p. 624)
- arithmetic sequence (p. 622)
- arithmetic series (p. 629)
- Binomial Theorem (p. 665)
- common difference (p. 622)
- common ratio (p. 636)
- convergent series (p. 651)
- factorial (p. 666)
- Fibonacci sequence (p. 658)
- geometric means (p. 638)
- geometric sequence (p. 636)
- geometric series (p. 643)
- index of summation (p. 631)
- inductive hypothesis
- infinite geometric series (p. 650)
- iteration (p. 660)
- mathematical induction (p. 670)
- partial sum (p. 650)
- Pascal's triangle (p. 664)
- recursive formula (p. 658)
- sequence (p. 622)
- series (p. 629)
- sigma notation (p. 631)
- term (p. 622)

Vocabulary Check

Choose the term from the list above that best completes each statement.

1. A(n) ________ of an infinite series is the sum of a certain number of terms.

2. If a sequence has a common ratio, then it is a(n) ________.

3. Using ________ , the series 2 + 5 + 8 + 11 + 14 can be written as \( \sum_{n=1}^{5} (3n - 1) \).

4. Eleven and 17 are two ________ between 5 and 23 in the sequence 5, 11, 17, 23.

5. Using the ________ , \((a - 2)^4\) can be expanded to \(a^4 - 8a^3 + 24a^2 - 32a + 16\).

6. The ________ of the sequence 3, 4, 8, 16, 27 is \(\frac{2}{3}\).

7. The ________ 11 + 16.5 + 22 + 27.5 + 33 has a sum of 110.

8. A(n) ________ is expressed as \(n! = n(n - 1)(n - 2) \ldots 2 \cdot 1\).
Lesson-by-Lesson Review

11-1

Arithmetic Sequences (pp. 622-628)

Find the indicated term of each arithmetic sequence.
9. \(a_1 = 6, d = 8, n = 5\)
10. \(a_1 = -5, d = 7, n = 22\)
11. \(a_1 = 5, d = -2, n = 9\)
12. \(a_1 = -2, d = -3, n = 15\)

Find the arithmetic means in each sequence.
13. \(-7, \_, \_, \_, \_, 9\)
14. \(12, \_, \_, \_, \_, 4\)
15. \(9, \_, \_, \_, \_, \_, -6\)
16. \(56, \_, \_, \_, \_, 28\)

17. GLACIERS The fastest glacier is recorded to have moved 12 kilometers every three months. If the glacier moved at a constant speed, how many kilometers did it move in one year?

Example 1 Find the 12th term of an arithmetic sequence if \(a_1 = -17\) and \(d = 4\).
\[
a_n = a_1 + (n - 1)d
\]
Formula for the \(n\)th term
\[
a_{12} = -17 + (12 - 1)4 \quad n = 12, a_1 = -17, d = 4
\]
\[
a_{12} = 27
\]
Simplify.

Example 2 Find the two arithmetic means between 4 and 25.
\[
a_n = a_1 + (n - 1)d
\]
Formula for the \(n\)th term
\[
a_4 = 4 + (4 - 1)d \quad n = 4, a_1 = 4
\]
\[
25 = 4 + 3d \quad a_4 = 25
\]
\[
7 = d
\]
Simplify.
The arithmetic means are 4 + 7 or 11 and 11 + 7 or 18.

11-2

Arithmetic Series (pp. 629-635)

Find \(S_n\) for each arithmetic series.
18. \(a_1 = 12, a_n = 117, n = 36\)
19. \(4 + 10 + 16 + \ldots + 106\)
20. \(10 + 4 + (-2) + \ldots + (-50)\)
21. Evaluate \(\sum_{n=2}^{13} (3n + 1)\).
22. PATTERNS On the first night of a celebration, a candle is lit and then blown out. The second night, a new candle and the candle from the previous night are lit and blown out. This pattern of lighting a new candle and all the candles from the previous nights is continued for seven nights. Find the total number of candle lightings.

Example 3 Find \(S_n\) for the arithmetic series with \(a_1 = 34, a_n = 2\), and \(n = 9\).
\[
S_n = \frac{n}{2} (a_1 + a_n)
\]
Sum formula
\[
S_9 = \frac{9}{2} (34 + 2) \quad n = 9, a_1 = 34, a_n = 2
\]
\[
= 162
\]
Simplify.

Example 4 Evaluate \(\sum_{n=5}^{11} (2n - 3)\).
Use the formula \(S_n = \frac{n}{2} (a_1 + a_n)\). There are 7 terms, \(a_1 = 2(5) - 3\) or 7, and \(a_7 = 2(11) - 3\) or 19.
\[
S_7 = \frac{7}{2} (19 + 7)
\]
\[
= 91
\]
11–3 Geometric Sequences (pp. 636-641)

Find the indicated term of each geometric sequence.
23. \(a_1 = 2, r = 2, n = 5\)
24. \(a_1 = 7, r = 2, n = 4\)
25. \(a_1 = 243, r = -\frac{1}{3}, n = 5\)
26. \(a_6\) for \(\frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots\)

Find the geometric means in each sequence.
27. 3, \(\_\_\_, \_\_\_, 24\)
28. 7.5, \(\_\_\_, \_\_\_, \_\_\_, 120\)

29. SAVINGS Kathy has a savings account with a current balance of $5000. What would Kathy’s account balance be after five years if she receives 3% interest annually?

Example 5 Find the fifth term of a geometric sequence for which \(a_1 = 7\) and \(r = 3\).

\[
a_n = a_1 \cdot r^{n-1}
\]

Formula for the \(n\)th term

\[a_5 = 7 \cdot 3^{5-1} = 567\]

The fifth term is 567.

Example 6 Find two geometric means between 1 and 8.

\[
a_n = a_1 \cdot r^{n-1}
\]

Formula for the \(n\)th term

\[a_4 = 1 \cdot r^{4-1} = 8\]

\[8 = r^3\]

\[2 = r\] Simplify.

The geometric means are 1(2) or 2 and 2(2) or 4.

11–4 Geometric Series (pp. 643-649)

Find \(S_n\) for each geometric series.
30. \(a_1 = 12, r = 3, n = 5\)
31. \(4 - 2 + 1 - \ldots\) to 6 terms
32. \(256 + 192 + 144 + \ldots\) to 7 terms
33. Evaluate \(\sum_{n=1}^{5} \left(-\frac{1}{2}\right)^{n-1}\).

34. TELEPHONES Joe started a phone tree to give information about a party to his friends. Joe starts by calling 3 people. Then each of those 3 people calls 3 people, and each person who receives a call then calls 3 more people. How many people have been called after 4 layers of the phone tree? (Hint: Joe is considered the first layer.)

Example 7 Find the sum of a geometric series for which \(a_1 = 7, r = 3,\) and \(n = 14\).

\[
S_n = \frac{a_1 - a_1 r^n}{1 - r}
\]

Sum formula

\[S_{14} = \frac{7 - 7 \cdot 3^{14}}{1 - 3} = 16,740,388\] Use a calculator.

Example 8 Evaluate \(\sum_{n=1}^{5} \left(\frac{3}{4}\right)^{n-1}\).

\[
S_5 = \frac{1 - \left(\frac{3}{4}\right)^5}{1 - \frac{3}{4}}
\]

\[n = 5, a_1 = 1, r = \frac{3}{4}\]

\[S_5 = \frac{1 - \left(\frac{3}{4}\right)}{1 - \frac{3}{4}} = \frac{\frac{781}{1024}}{\frac{1}{4}} = \frac{3}{4} \cdot \frac{243}{1024} = \frac{781}{256}\]
11–5 Infinite Geometric Series (pp. 650-655)

Find the sum of each infinite geometric series, if it exists.

35. \(a_1 = 6, r = \frac{11}{12}\)

36. \(\frac{1}{8} - \frac{3}{16} + \frac{9}{32} - \frac{27}{64} + \cdots\)

37. \(\sum_{n=1}^{\infty} -\frac{5}{8}^{n-1}\)

38. GEOMETRY If the midpoints of the sides of \(\triangle ABC\) are connected, a smaller triangle results. Suppose the process of connecting midpoints of sides and drawing new triangles is continued indefinitely. Find the sum of the perimeters of all of the triangles if the perimeter of \(\triangle ABC\) is 30 centimeters.

Example 9 Find the sum of the infinite geometric series for which \(a_1 = 18\) and \(r = -\frac{2}{7}\).

\[S = \frac{a_1}{1 - r} \quad \text{Sum formula}\]

\[= \frac{18}{1 - \left(-\frac{2}{7}\right)}\]

\[= \frac{18}{9} \text{ or } 2\]

Simplify.

11–6 Recursion and Special Sequences (pp. 658-662)

Find the first five terms of each sequence.

39. \(a_1 = -2, a_{n+1} = a_n + 5\)

40. \(a_1 = 3, a_{n+1} = 4a_n - 10\)

Find the first three iterates of each function for the given initial value.

41. \(f(x) = -2x + 3, x_0 = 1\)

42. \(f(x) = 7x - 4, x_0 = 2\)

43. SAVINGS Toni has a savings account with a $15,000 balance. She has a 4% interest rate that is compounded monthly. Every month Toni makes a $1000 withdrawal from the account to cover her expenses. The recursive formula \(b_n = 1.04b_{n-1} - 1000\) describes the balance in Toni’s savings account after \(n\) months. Find the balance of Toni’s account after the first four months. Round your answer to the nearest dollar.
Chapter 11

Study Guide and Review

11–7 The Binomial Theorem (pp. 664-669)

Expand each power.

44. \((x - 2)^4\) 45. \((3r + s)^5\)

Find each indicated term of each expansion.

46. fourth term of \((x + 2y)^6\)
47. second term of \((4x - 5)^{10}\)

48. SCHOOL Mr. Brown is giving a four-question multiple-choice quiz. Each question can be answered A, B, C, or D. How many ways could a student answer the questions using each answer A, B, C, or D once?

Example 11 Expand \((a - 2b)^4\).

\[
(a - 2b)^4 = \sum_{k=0}^{4} \frac{4!}{(4-k)!k!} a^{4-k}(-2b)^k
\]

\[
= \frac{4!}{0!4!} a^4(-2b)^0 + \frac{4!}{3!1!} a^3(-2b)^1 + \frac{4!}{2!2!} a^2(-2b)^2 + \frac{4!}{1!3!} a^1(-2b)^3 + \frac{4!}{0!4!} a^0(-2b)^4
\]

\[
= a^4 - 8a^3b + 24a^2b^2 - 32ab^3 + 16b^4
\]

11–8 Proof and Mathematical Induction (pp. 670-674)

Prove that each statement is true for all positive integers.

49. \(1 + 2 + 4 + \ldots + 2^n - 1 = 2^n - 1\)
50. \(6^n - 1\) is divisible by 5.
51. \(3^n - 1\) is divisible by 2.
52. \(1 + 4 + 7 + \ldots + (3n - 2) = \frac{n(3n - 1)}{2}\)

Find a counterexample for each statement.

53. \(n^2 - n + 13\) is prime.
54. \(13^n + 11\) is divisible by 24.
55. \(9^n + 1\) is divisible by 16.
56. \(n^2 + n + 1\) is prime.

Example 12 Prove that \(1 + 5 + 25 + \ldots + 5^{n-1} = \frac{1}{4}(5^n - 1)\) for positive integers \(n\).

Step 1 When \(n = 1\), the left side of the given equation is 1. The right side is \(\frac{1}{4}(5^1 - 1)\) or 1. Thus, the equation is true for \(n = 1\).

Step 2 Assume that \(1 + 5 + 25 + \ldots + 5^{k-1} = \frac{1}{4}(5^k - 1)\) for some positive integer \(k\).

Step 3 Show that the given equation is true for \(n = k + 1\).

\[
1 + 5 + 25 + \ldots + 5^{k-1} + 5^k - 1
= \frac{1}{4}(5^k - 1) + 5^{k+1} - 1
\]

Add to each side.

\[
= \frac{1}{4}(5^k - 1) + 5^k
\]

Simplify the exponent.

\[
= \frac{5^k - 1 + 4 \cdot 5^k}{4}
\]

Common denominator

\[
= \frac{5 \cdot 5^k - 1}{4}
\]

Distributive Property

\[
= \frac{1}{4}(5^{k+1} - 1)
\]

Thus, the equation is true for \(n = k + 1\). The conjecture is proved.
1. Find the next four terms of the arithmetic sequence 42, 37, 32, ….

2. Find the 27th term of an arithmetic sequence for which \( a_1 = 2 \) and \( d = 6 \).

3. **MULTIPLE CHOICE** What is the tenth term in the arithmetic sequence that begins 10, 5.6, 1.2, -3.2, …?

   A. -39.6
   B. -29.6
   C. 29.6
   D. 39.6

4. Find the three arithmetic means between -4 and 16.

5. Find the sum of the arithmetic series for which \( a_1 = 7 \), \( n = 31 \), and \( a_n = 127 \).

6. Find the next two terms of the geometric sequence \( \frac{1}{81}, \frac{1}{27}, \frac{1}{9}, \ldots \).

7. Find the sixth term of the geometric sequence for which \( a_1 = 5 \) and \( r = -2 \).

8. **MULTIPLE CHOICE** Find the next term in the geometric sequence 8, 6, \( \frac{9}{2}, \frac{27}{8}, \ldots \).

   F. \( \frac{11}{8} \)
   G. \( \frac{27}{16} \)
   H. \( \frac{9}{4} \)
   J. \( \frac{81}{32} \)

9. Find the two geometric means between 7 and 189.

10. Find the sum of the geometric series for which \( a_1 = 125 \), \( r = \frac{2}{5} \), and \( n = 4 \).

**Find the sum of each series, if it exists.**

11. \( \sum_{k=3}^{15} (14 - 2k) \)

12. \( \sum_{n=1}^{\infty} \frac{1}{3}(-2)^n - 1 \)

13. \( 91 + 85 + 79 + \cdots + (-29) \)

14. \( 12 + (-6) + 3 + \left( -\frac{3}{2} \right) + \cdots \)

**Find the first five terms of each sequence.**

15. \( a_1 = 1, a_{n+1} = a_n + 3 \)

16. \( a_1 = -3, a_{n+1} = a_n + n^2 \)

17. Find the first three iterates of \( f(x) = x^2 - 3x \) for an initial value of \( x_0 = 1 \).

18. Expand \( (2s - 3t)^5 \).

19. What is the coefficient of the fifth term of \( (r + 2q)^7 \)?

20. Find the third term of the expansion of \( (x + y)^{10} \).

**Prove that each statement is true for all positive integers.**

21. \( 1 + 7 + 49 + \cdots + 7^n - 1 = \frac{1}{6}(7^n - 1) \)

22. \( 14^n - 1 \) is divisible by 13.

23. Find a counterexample for the following statement.

   *The units digit of \( 7^n - 3 \) is never 8.*

24. **DESIGN** The pattern in a red and white brick wall starts with 20 red bricks on the bottom row. Each row contains 3 fewer red bricks than the row below. If the top row has no red bricks, how many rows are there and how many red bricks were used?

25. **RECREATION** One minute after it is released, a gas-filled balloon has risen 100 feet. In each succeeding minute, the balloon rises only 50% as far as it rose in the previous minute. How far will it rise in 5 minutes?
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. How many 3-inch cubes can be placed completely inside a box that is 15 inches long, 12 inches wide, and 18 inches tall?
   A 5
   B 20
   C 120
   D 360

2. Using the table below, which expression can be used to determine the nth term of the sequence?

   \[
   \begin{array}{c|c}
   n & y \\
   \hline
   1 & 6 \\
   2 & 10 \\
   3 & 14 \\
   4 & 18 \\
   \end{array}
   \]

   F \( y = 6n \)
   G \( y = n + 5 \)
   H \( y = 2n + 1 \)
   J \( y = 2(2n + 1) \)

3. **GRIDDABLE** The pattern of squares below continues infinitely, with more squares being added at each step. How many squares are in the tenth step?

4. The pattern of dots shown below continues infinitely, with more dots being added at each step.

   \[
   \begin{array}{ccc}
   \cdot & \cdot & \cdot \\
   \cdot & \cdot & \cdot & \cdot \\
   \cdot & \cdot & \cdot & \cdot & \cdot \\
   \end{array}
   \]

   Which expression can be used to determine the number of dots in the nth step?
   A \( 2n \)
   B \( n(n + 2) \)
   C \( n(n + 1) \)
   D \( 2(n + 1) \)

5. The figures below show a pattern of dark tiles and white tiles that can be described by a relationship between two variables.

   \[
   \begin{array}{c}
   \square \\
   \begin{array}{c}
   \begin{array}{c}
   \square \square \\
   \square \square \square \\
   \end{array}
   \end{array}
   \end{array}
   \]

   Which rule relates \( d \), the number of dark tiles, to \( w \), the number of white tiles?
   F \( d = 2w \)
   G \( w = d - 1 \)
   H \( d = 2w - 2 \)
   J \( w = \frac{1}{2}d + 1 \)

6. Leland is renting an apartment. He looked at a 3-bedroom apartment for $950 per month near the downtown area, and a 3-bedroom apartment for $725 per month on the edge of town. About what percent of the cost of the downtown apartment is Leland saving by renting the apartment on the edge of town?
   A 2%
   B 24%
   C 31%
   D 231%
7. What is the volume of a 3-dimensional object with the dimensions shown in the 3 views below?

F  864 in³  
G  1056 in³  
H  1248 in³  
J  1440 in³  

8. △ABC is graphed on the coordinate grid below.

Which set of coordinates represents the vertices of a triangle congruent to △ABC?
A (−1, 7), (−1, 15), (3, 8)  
B (2, 7), (2, 14), (3, 7)  
C (4, 7), (4, 14), (7, 7)  
D (−1, 7), (−1, 14), (3, 7)  

9. The radius of the larger sphere shown below was multiplied by a factor of $\frac{1}{3}$ to produce the smaller sphere.

How does the volume of the smaller sphere compare to the volume of the larger sphere?
F The volume of the smaller sphere is $\frac{1}{9}$ as large.  
G The volume of the smaller sphere is $\frac{1}{\pi^3}$ as large.  
H The volume of the smaller sphere is $\frac{1}{27}$ as large.  
J The volume of the smaller sphere is $\frac{1}{3}$ as large.  

10. GRIDDABLE Marla is putting a binding around a square quilt. The length of the binding was 32 feet. Find the approximate length, in feet, of the diagonal of the square quilt. Round to one decimal place.

Pre-AP

Record your answers on a sheet of paper. Show your work.

11. Kyla’s annual salary is $50,000. Each year she gets a 6% raise.

a. To the nearest dollar, what will her salary be in four years?  

b. To the nearest dollar, what will her salary be in 10 years?