Many real-world situations can be modeled using linear equations. But there are also many situations for which a linear equation would not be an accurate model. The power generated by a windmill can be best modeled using a polynomial function.

Key Vocabulary
- **polynomial function** (p. 332)
- **scientific notation** (p. 315)
- **synthetic division** (p. 327)
- **synthetic substitution** (p. 356)

Real-World Link
**Power Generation** Many real-world situations can be modeled using linear equations. But there are also many situations for which a linear equation would not be an accurate model. The power generated by a windmill can be best modeled using a polynomial function.
Option 2

Take the Online Readiness Quiz at algebra2.com.

Option 1

Take the Quick Check below. Refer to the Quick Review for help.

Rewrite each difference as a sum (Prerequisite Skill)

1. $2 - 7$
2. $-6 - 11$
3. $x - y$
4. $8 - 2x$
5. $2xy - 6yz$
6. $6a^2b - 12b^2c$

7. **CANDY** Janet has $4. She buys $x$ candy bars for $0.50 each. Rewrite the amount of money she has left as a sum. (Prerequisite Skill)

Use the Distributive Property to rewrite each expression without parentheses. (Lesson 1-2)

8. $-2(4x^3 + x - 3)$
9. $-1(x + 2)$
10. $-1(x - 3)$
11. $-3(2x^4 - 5x^2 - 2)$
12. $-rac{1}{2}(3a + 2)$
13. $-rac{2}{3}(2 + 6z)$

**SCHOOL SHOPPING** For Exercises 14 and 15, use the following information.

Students, ages 12 to 17, plan on spending an average of $113 on clothing for school. The students plan on spending 36% of their money at specialty stores and 19% at department stores. (Lesson 1-2)

14. Write an expression to represent the amount that the average student spends shopping for clothes at specialty and department stores.

15. Evaluate the expression from Exercise 14 by using the Distributive Property.

Solve each equation. (Lesson 5-6)

16. $x^2 - 17x + 60 = 0$
17. $14x^2 + 23x + 3 = 0$
18. $2x^2 + 5x + 1 = 0$
19. $3x^2 - 5x + 2 = 0$
Properties of Exponents

Economists often deal with very large numbers. For example, the table shows the U.S. public debt for several years. Such numbers, written in standard notation, are difficult to work with because they contain so many digits. Scientific notation uses powers of ten to make very large or very small numbers more manageable.

Multiply and Divide Monomials To simplify an expression containing powers means to rewrite the expression without parentheses or negative exponents. Negative exponents are a way of expressing the multiplicative inverse of a number. For example, \( \frac{1}{x^2} \) can be written as \( x^{-2} \). Note that an expression such as \( x^{-2} \) is not a monomial. Why?

**KEY CONCEPT**

### Negative Exponents

**Words** For any real number \( a \neq 0 \) and any integer \( n \), \( a^{-n} = \frac{1}{a^n} \) and \( \frac{1}{a^{-n}} = a^n \).

**Examples** \( 2^{-3} = \frac{1}{2^3} \) and \( \frac{1}{b^{-8}} = b^8 \)

**EXAMPLE** Simplify Expressions with Multiplication

Simplify each expression. Assume that no variable equals 0.

a. \( (3x^3y^2)(-4x^2y^4) \)

\[
(3x^3y^2)(-4x^2y^4) = (3 \cdot x \cdot x \cdot x \cdot y \cdot y) \cdot (-4 \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y) \\
= 3(-4) \cdot x \cdot x \cdot x \cdot x \cdot y \cdot y \cdot y \cdot y \cdot y \\
= -12x^5y^6
\]
Lesson 6-1  Properties of Exponents

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Example 1 suggests the following property of exponents.

Example 1

Simplify Expressions with Division

Simplify \( \frac{p^3}{p^8} \). Assume that \( p \neq 0 \).

\[
\frac{p^3}{p^8} = p^{3-8} = p^{-5} \quad \text{or} \quad \frac{1}{p^5}
\]

Remember that a simplified expression cannot contain negative exponents.

Check Your Progress

Simplify each expression. Assume that no variable equals 0.

2A. \( \frac{y^{12}}{y^4} \)

2B. \( \frac{15c^5d^3}{-3c^2d^7} \)

Extra Examples at algebra2.com
You can use the Quotient of Powers property and the definition of exponents to simplify \( \frac{y^4}{y^3} \) if \( y \neq 0 \).

**Method 1**

\[
\frac{y^4}{y^3} = y^{4-3} \quad \text{Quotient of Powers} \\
= y \quad \text{Subtract.}
\]

**Method 2**

\[
\frac{y^4}{y^3} = \frac{y \cdot y \cdot y \cdot y}{y \cdot y \cdot y} \quad \text{Definition of exponents} \\
= 1 \quad \text{Divide.}
\]

In order to make the results of these two methods consistent, we define \( y^0 = 1 \), where \( y \neq 0 \). In other words, any nonzero number raised to the zero power is equal to 1. \( \text{Notice that } 0^0 \text{ is undefined.} \)

The properties we have presented can be used to verify the properties of powers that are listed below.

<table>
<thead>
<tr>
<th><strong>Words</strong></th>
<th><strong>Examples</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Suppose ( a ) and ( b ) are real numbers and ( m ) and ( n ) are integers. Then the following properties hold.</td>
<td>[(a^2)^3 = a^6]</td>
</tr>
<tr>
<td>Power of a Power: ( (a^m)^n = a^{mn} )</td>
<td>[(xy)^2 = x^2y^2]</td>
</tr>
<tr>
<td>Power of a Product: ( (ab)^m = a^m b^m )</td>
<td>[\left(\frac{a}{b}\right)^3 = \frac{a^3}{b^3}]</td>
</tr>
<tr>
<td>Power of a Quotient: ( \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n} ), ( b \neq 0 ) and ( \left(\frac{b}{a}\right)^{-n} = \left(\frac{b}{a}\right)^n ) or ( b^n \cdot a^{-n} ), ( a \neq 0 ), ( b \neq 0 )</td>
<td>[\left(\frac{x}{y}\right)^{-4} = \frac{y^4}{x^4}]</td>
</tr>
<tr>
<td>Zero Power: ( a^0 = 1 ), ( a \neq 0 )</td>
<td>[2^0 = 1]</td>
</tr>
</tbody>
</table>

**Example**

Simplify each expression.

**a.** \((a^3)^6\)

\[
(a^3)^6 = a^{3(6)} \quad \text{Power of a power} \\
= a^{18} \quad \text{Simplify.}
\]

**b.** \((-3x)^4 \div y^4\)

\[
\frac{(-3x)^4}{y^4} = \frac{(-3x)^4}{y^4} \quad \text{Power of a quotient} \\
= \frac{(-3)^4x^4}{y^4} \quad \text{Power of a product} \\
= \frac{81x^4}{y^4} \quad (\text{-}3)^4 = 81
\]

With complicated expressions, you often have a choice of which way to start simplifying.
EXAMPLE Simplify Expressions Using Several Properties

4 Simplify \( \left( \frac{-2x^{3n}}{x^{2n}y^3} \right)^4 \).

Method 1
Raise the numerator and denominator to the fourth power before simplifying.

\[
\left( \frac{-2x^{3n}}{x^{2n}y^3} \right)^4 = \left( \frac{-2x^{3n}}{x^{2n}y^3} \right)^4
\]

\[
= (-2)^4(x^{3n})^4
\]

\[
= 16x^{12n}
\]

\[
\frac{y^{-12}}{y^{12}} = \frac{16x^{12n-8n}}{y^{12}} = 16x^{4n}
\]

Method 2
Simplify the fraction before raising to the fourth power.

\[
\left( \frac{-2x^{3n}}{x^{2n}y^3} \right)^4 = \left( \frac{-2x^{3n} - 2n}{y^3} \right)^4
\]

\[
= \frac{(-2)^4(y^3)^4}{y^{12}}
\]

\[
= 16x^{4n}
\]

\[
\frac{y^{12}}{y^{12}} = \frac{16x^{4n}}{y^{12}}
\]

4A. \( \left( \frac{3x^2y^3}{2y^4} \right)^3 \)

4B. \( \left( \frac{-3x^{-5}y^{-2n}}{5x^{-6}} \right)^4 \)

CHECK Your Progress

Scientific Notation The form that you usually write numbers in is standard notation. A number is in scientific notation when it is in the form \( a \times 10^n \), where \( 1 \leq a < 10 \) and \( n \) is an integer. Real-world problems using numbers in scientific notation often involve units of measure. Performing operations with units is known as dimensional analysis.

Real-World EXAMPLE ASTRONOMY After the Sun, the next-closest star to Earth is Alpha Centauri C, which is about \( 4 \times 10^{16} \) meters away. How long does it take light from Alpha Centauri C to reach Earth? Use the information at the left.

Begin with the formula \( d = rt \), where \( d \) is distance, \( r \) is rate, and \( t \) is time.

\[
t = \frac{d}{r} \quad \text{Solve the formula for time.}
\]

\[
= \frac{4 \times 10^{16} \text{ m}}{3.00 \times 10^8 \text{ m/s}} \quad \text{Distance from Alpha Centauri C to Earth}
\]

\[
= \frac{4}{3.00} \cdot \frac{10^{16}}{10^8} \cdot \frac{\text{m}}{\text{m/s}} = 1.33 \times 10^8 \text{ s}
\]

Estimate: The result should be slightly greater than \( \frac{10^{16}}{10^8} \) or \( 10^{16} \).

\[
\approx 1.33 \times 10^8 \text{ s}
\]

It takes about \( 1.33 \times 10^8 \) seconds or 4.2 years for light from Alpha Centauri C to reach Earth.

CHECK Your Progress

5. The density \( D \) of an object in grams per milliliter is found by dividing the mass \( m \) of the substance by the volume \( V \) of the object. A sample of platinum has a mass of \( 8.4 \times 10^{-2} \) kilogram and a volume of \( 4 \times 10^{-6} \) cubic meter. Use this information to calculate the density of platinum.
Exercises

Simplify. Assume that no variable equals 0.

1. \((-3x^2y^3)(5x^5y^6)\)
2. \(\frac{30y^4}{-5y^2}\)
3. \(-\frac{2a^3b^6}{18a^2b^2}\)
4. \((2b)^4\)
5. \(\left(\frac{1}{w^{4z^2}}\right)^3\)
6. \(\left(\frac{cd}{3}\right)^{-2}\)
7. \((n^3)(n^{-3})^3\)
8. \(\frac{81p^6q^5}{(3p^2q)^2}\)
9. \(-\frac{6x^6}{3x^3}^{-2}\)

10. **ASTRONOMY** Refer to Example 5 on page 315. The average distance from Earth to the Moon is about \(3.84 \times 10^8\) meters. How long would it take a radio signal traveling at the speed of light to cover that distance?

**HELP** See Examples (p. 315)

**POLYNOMIAL FUNCTIONS**

Simplify. Assume that no variable equals 0.

11. \(\frac{1}{3}a^8b^2(2a^2b)\)
12. \((5cd^2)(-c^4d)\)
13. \((7x^3y^{-5})(4xy^2)\)
14. \((-3b^3)(7b^2c^2)\)
15. \(\frac{a^2n^6}{an^5}\)
16. \(-\frac{y^5z^7}{y^2z^5}\)
17. \(-5x^3y^3z^4\)
18. \(\frac{3ab^3c^3}{9a^3b^7c}\)
19. \((n^4)^4\)
20. \((z^2)^5\)
21. \((2x)^4\)
22. \((-2c)^3\)
23. \((a^3b)^3(ab)^{-2}\)
24. \((-2r^2s)^3(3rs^2)\)
25. \(\frac{2c^3d(3c^2d^5)}{30c^4d^2}\)
26. \(-\frac{12m^4n^6 (m^3n^2)}{36m^3n}\)

**BIOLOGY** Use the diagram at the right to write the diameter of a typical flu virus in scientific notation. Then estimate the area of a typical flu virus. *(Hint: Treat the virus as a circle.)*

27. The population of Earth is about \(6.445 \times 10^9\). The land surface area of Earth is \(1.483 \times 10^8\) km\(^2\). What is the population density for the land surface area of Earth?

**POPULATION**

Simplify. Assume that no variable equals 0.

29. \(2x^2(6y^3)(2x^2y)\)
30. \(3a(5a^2b)(6ab^3)\)
31. \(\frac{12x^{-3}y^{-2}z^{-8}}{30x^{-6}y^{-4}z^{-1}}\)
32. \(\frac{3a(5a^2b)(6ab^3)}{3a(5a^2b)(6ab^3)}\)
33. \(\left(\frac{x}{y^1}\right)^{-2}\)
34. \(\left(\frac{v}{w^2}\right)^{-3}\)
35. \(\frac{8a^2b^2}{16a^2b^2}\)
36. \(\frac{6a^2y^4}{3a^2y^3}\)
37. \(\frac{4x^{-3}y^2}{xy^{-5}}\)

**HELP** See Examples (p. 315)
38. If $2^r + 5 = 2^{2r} - 1$, what is the value of $r$?
39. What value of $r$ makes $y^{28} = y^{3r} \cdot y^7$ true?

40. **INCOME** In 2003, the population of Texas was about $2.21 \times 10^7$. The personal income for the state that year was about $6.43 \times 10^{11}$ dollars. What was the average personal income?

41. **RESEARCH** Use the Internet or other source to find the masses of Earth and the Sun. About how many times as large as Earth is the Sun?

42. **OPEN ENDED** Write an example that illustrates a property of powers. Then use multiplication or division to explain why it is true.

43. **FIND THE ERROR** Alejandra and Kyle both simplified $\frac{2a^2b}{(-2ab^3)^2}$. Who is correct? Explain your reasoning.

44. **REASONING** Determine whether $x^y \cdot x^z = x^{yz}$ is sometimes, always, or never true. Explain your reasoning.

45. **CHALLENGE** Determine which is greater, $100^{10}$ or $10^{100}$. Explain.

46. **Writing in Math** Use the information on page 312 to explain why scientific notation is useful in economics. Include the 2004 national debt of $7,379,100,000,000 and the U.S. population of 293,700,000, both written in words and in scientific notation, and an explanation of how to find the amount of debt per person with the result written in scientific notation and in standard notation.

47. **ACT/SAT** Which expression is equal to $\frac{(2x^3)^3}{12x^4}$?

   A $\frac{x}{2}$
   B $\frac{2x}{3}$
   C $\frac{1}{2x^2}$
   D $\frac{2x^2}{3}$

48. **REVIEW** Four students worked the same math problem. Each student’s work is shown below.

   **Student F**
   
   $x^2 x^{-5} = \frac{x^2}{x^5} = \frac{1}{x^3}, \ x \neq 0$

   **Student G**
   
   $x^2 x^{-5} = \frac{x^2}{x^5} = x^{-3}, \ x \neq 0$

   **Student H**
   
   $x^2 x^{-5} = \frac{x^2}{x^{-5}} = x^7, \ x \neq 0$

   **Student J**
   
   $x^2 x^{-5} = \frac{x^2}{x^{-5}} = x^7, \ x \neq 0$

   Which is a completely correct solution?

   F Student F  G Student G  H Student H  J Student J
Solve each inequality algebraically.  (Lesson 5-8)

49. \( x^2 - 8x + 12 < 0 \)  
50. \( x^2 + 2x - 86 \geq -23 \)  
51. \( 15x^2 + 4x + 12 \leq 0 \)

Graph each function.  (Lesson 5-7)

52. \( y = -2(x - 2)^2 + 3 \)  
53. \( y = \frac{1}{3}(x + 5)^2 - 1 \)  
54. \( y = \frac{1}{2}x^2 + x + \frac{3}{2} \)

Evaluate each determinant.  (Lesson 4-3)

55. \[
\begin{vmatrix}
3 & 0 \\
2 & -2
\end{vmatrix}
\]
56. \[
\begin{vmatrix}
1 & 0 & -3 \\
2 & -1 & 4 \\
-3 & 0 & 2
\end{vmatrix}
\]

Solve each system of equations.  (Lesson 3-5)

57. \( x + y = 5 \)  
58. \( a + b + c = 6 \)

\[ x + y + z = 4 \]  
\[ 2a - b + 3c = 16 \]

\[ 2x - y + 2z = -1 \]  
\[ a + 3b - 2c = -6 \]

Identify each function as S for step, C for constant, A for absolute value, or P for piecewise.  (Lesson 2-6)

TRANSPORTATION  For Exercises 62–64, refer to the graph at the right.  (Lesson 2-5)

62. Make a scatter plot of the data, where the horizontal axis is the number of years since 1975.
63. Write a prediction equation.
64. Predict the median age of vehicles on the road in 2015.

Solve each equation.  (Lesson 1-3)

65. \( 2x + 11 = 25 \)  
66. \( -12 - 5x = 3 \)

PREREQUISITE SKILL  Use the Distributive Property to find each product.  (Lesson 1-2)

67. \( 2(x + y) \)  
68. \( 3(x - z) \)  
69. \( 4(x + 2) \)

70. \( -2(3x - 5) \)  
71. \( -5(x - 2y) \)  
72. \( -3(-y + 5) \)

Median Age of Vehicles

<table>
<thead>
<tr>
<th>Year</th>
<th>Median Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>1970</td>
<td>4.9 years</td>
</tr>
<tr>
<td>1975</td>
<td>5.4 years</td>
</tr>
<tr>
<td>1980</td>
<td>6 years</td>
</tr>
<tr>
<td>1985</td>
<td>6.9 years</td>
</tr>
<tr>
<td>1990</td>
<td>6.5 years</td>
</tr>
<tr>
<td>1995</td>
<td>7.7 years</td>
</tr>
<tr>
<td>1999</td>
<td>8.3 years</td>
</tr>
<tr>
<td>2004</td>
<td>8.9 years</td>
</tr>
</tbody>
</table>

Source: Transportation Department
Dimensional Analysis

Real-world problems often involve units of measure. Performing operations with units is called **dimensional analysis**. You can use dimensional analysis to convert units or to perform calculations.

**Example**  A car’s gas tank holds 14 gallons of gasoline and the car gets 16 miles per gallon. How many miles can be driven on a full tank of gasoline?

You want to find the number of miles that can be driven on 1 tank of gasoline, or the number of miles per tank. You know that there are 14 gallons per tank and 16 miles per gallon. Translate these into fractions that you can multiply.

\[
\frac{14 \text{ gal}}{1 \text{ tank}} \times \frac{16 \text{ mi}}{1 \text{ gal}} = \frac{14 \text{ gal}}{1 \text{ tank}} \times \frac{16 \text{ mi}}{1 \text{ gal}}
\]

The units **gallons** cancel out.

\[
= (14)(16) \text{ mi/tank}
\]

Simplify.

\[
= 224 \text{ mi/tank}
\]

Multiply.

So, 224 miles can be driven on a full tank of gasoline. This answer is reasonable because the final units are mi/tank, not mi/gal, gal/mi, or mi.

**Reading to Learn**

Solve each problem using dimensional analysis. Include the appropriate units with your answer.

1. How many miles will a person run during a 5-kilometer race?  
   
   (Hint: 1 km \( \approx \) 0.62 mi)

2. A zebra can run 40 miles per hour. How far can a zebra run in 3 minutes?

3. A cyclist traveled 43.2 miles at an average speed of 12 miles per hour. How long did the cyclist ride?

4. The average student is in class 315 minutes/day. How many hours per day is this?

5. If you are going 50 miles per hour, how many feet per second are you traveling?

6. The equation \( d = \frac{1}{2}(9.8 \text{ m/s}^2)(3.5 \text{ s})^2 \) represents the distance \( d \) that a ball falls 3.5 seconds after it is dropped from a tower. Find the distance.

7. Explain what the following statement means.  
   
   **Dimensional analysis tells you what to multiply or divide.**

8. Explain how dimensional analysis can be useful in checking the reasonableness of your answer.
Shenequa has narrowed her choice for which college to attend. She is most interested in Coastal Carolina University, where the current year’s tuition is $3430. Shenequa assumes that tuition will increase at a rate of 6% per year. You can use polynomials to represent the increasing tuition costs.

**Add and Subtract Polynomials** If \( r \) represents the rate of increase of tuition, then the tuition for the second year will be \( 3430(1 + r) \). For the third year, it will be \( 3430(1 + r)^2 \), or \( 3430r^2 + 6860r + 3430 \) in expanded form. The **degree of a polynomial** is the degree of the monomial with the greatest degree. For example, the degree of this polynomial is 2.

**EXAMPLE**

**Degree of a Polynomial**

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

1. \( \frac{1}{6}x^3y^5 - 9x^4 \)
   - This expression is a polynomial because each term is a monomial. The degree of the first term is \( 3 + 5 \) or 8, and the degree of the second term is 4. The degree of the polynomial is 8.

2. \( x + \sqrt{x} + 5 \)
   - This expression is not a polynomial because \( \sqrt{x} \) is not a monomial.

3. \( x^{-2} + 3x^{-1} - 4 \)
   - This expression is not a polynomial because \( x^{-2} \) and \( x^{-1} \) are not monomials. \( x^{-2} = \frac{1}{x^2} \) and \( x^{-1} = \frac{1}{x} \). Monomials cannot contain variables in the denominator.

**CHECK Your Progress**

1A. \( \frac{x}{y} + 3x^2 \)  
1B. \( x^5y + 9x^4y^3 - 2xy \)

To *simplify* a polynomial means to perform the operations indicated and combine like terms.
Lesson 6-2 Operations with Polynomials

**EXAMPLE** Simplify Polynomials

Simplify each expression.

a. \((3x^2 - 2x + 3) - (x^2 + 4x - 2)\)

Remove parentheses and group like terms together.

\[
(3x^2 - 2x + 3) - (x^2 + 4x - 2) = 3x^2 - 2x + 3 - x^2 - 4x + 2
\]

Distribute the \(-1\).

\[
= (3x^2 - x^2) + (-2x - 4x) + (3 + 2)
\]

Combine like terms.

\[
= 2x^2 - 6x + 5
\]

b. \((5x^2 - 4x + 1) + (-3x^2 + x - 3)\)

Align like terms vertically and add.

\[
\begin{align*}
5x^2 - 4x + 1 \\
+(-3x^2 + x - 3) \\
\hline
2x^2 - 3x - 2
\end{align*}
\]

**CHECK Your Progress**

2A. \((-x^2 - 3x + 4) - (x^2 + 2x + 5)\)  
2B. \((3x^2 - 6) + (-x + 1)\)

**Multiply Polynomials** You can use the Distributive Property to multiply polynomials.

**EXAMPLE** Simplify Using the Distributive Property

Find \(2x(7x^2 - 3x + 5)\).

\[
2x(7x^2 - 3x + 5) = 2x(7x^2) + 2x(-3x) + 2x(5)
\]

Distribute the \(2x\).

\[
= 14x^3 - 6x^2 + 10x
\]

**CHECK Your Progress**

Find each product.

3A. \(\frac{4}{3}x^2(6x^2 + 9x - 12)\)  
3B. \(-2a(-3a^2 - 11a + 20)\)

You can use algebra tiles to model the product of two binomials.

**ALGEBRA LAB**

**Multiplying Binomials**

Use algebra tiles to find the product of \(x + 5\) and \(x + 2\).

- Draw a 90° angle on your paper.
- Use an \(x\) tile and a 1 tile to mark off a length equal to \(x + 5\) along the top.
- Use the tiles to mark off a length equal to \(x + 2\) along the side.
- Draw lines to show the grid formed.
- Fill in the lines with the appropriate tiles to show the area product. The model shows the polynomial \(x^2 + 7x + 10\).

The area of the rectangle is the product of its length and width. So, \((x + 5)(x + 2) = x^2 + 7x + 10\).
Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

1. \( a^2 + 5b \)
2. \( \frac{1}{3}x^3 - 9y \)
3. \( \frac{mz^2 - 3}{nz^3 + 1} \)

Simplify.

4. \( (2a + 3b) + (8a - 5b) \)
5. \( (x^2 - 4x + 3) - (4x^2 + 3x - 5) \)
6. \( 2x(3y + 9) \)
7. \( 2p^2q(5pq - 3p^3q^2 + 4pq^4) \)
8. \( (y - 10)(y + 7) \)
9. \( (x + 6)(x + 3) \)
10. \( (2z - 1)(2z + 1) \)
11. \( (2m - 3n)^2 \)
12. \( (x + 1)(x^2 - 2x + 3) \)
13. \( (2x - 1)(x^2 - 4x + 4) \)

14. **GEOMETRY** Find the area of the triangle.

![Triangle diagram]

\[ \text{Area} = \frac{1}{2} 	imes \text{base} 	imes \text{height} \]

\[ \text{base} = 5x \text{ ft} \]
\[ \text{height} = 3x + 5 \text{ ft} \]

\[ \text{Area} = \frac{1}{2} \times 5x \times (3x + 5) \]

Determine whether each expression is a polynomial. If it is a polynomial, state the degree of the polynomial.

15. \( 3z^2 - 5z + 11 \)
16. \( x^3 - 9 \)
17. \( \frac{6xy}{z} - \frac{3c}{d} \)
18. \( \sqrt{m - 5} \)
19. \( 5x^2y^4 + x\sqrt{3} \)
20. \( \frac{4}{3}y^2 + \frac{5}{6}y^7 \)
Simplify.

21. \((3x^2 - x + 2) + (x^2 + 4x - 9)\)
22. \((5y + 3y^2) + (-8y - 6y^2)\)
23. \((9r^2 + 6r + 16) - (8r^2 + 7r + 10)\)
24. \((7m^2 + 5m - 9) + (3m^2 - 6)\)
25. \(4b(cb - zd)\)
26. \(4a(3a^2 + b)\)
27. \(-5ab^2(-3a^2b + 6a^3b - 3a^4b^4)\)
28. \(2xy(3xy^3 - 4xy + 2y^4)\)
29. \((p + 6)(p - 4)\)
30. \((a + 6)(a + 3)\)
31. \((b + 5)(b - 5)\)
32. \((6 - z)(6 + z)\)
33. \((3x + 8)(2x + 6)\)
34. \((4y - 6)(2y + 7)\)
35. \((3b - c)^3\)
36. \((x^2 + xy + y^2)(x - y)\)

37. **PERSONAL FINANCE** Toshiro has $850 to invest. He can invest in a savings account that has an annual interest rate of 1.7%, and he can invest in a money market account that pays about 3.5% per year. Write a polynomial to represent the amount of interest he will earn in 1 year if he invests \(x\) dollars in the savings account and the rest in the money market account.

38. **E-SALES** For Exercises 38 and 39, use the following information.
A small online retailer estimates that the cost, in dollars, associated with selling \(x\) units of a particular product is given by the expression \(0.001x^2 + 5x + 500\). The revenue from selling \(x\) units is given by \(10x\).

- **38.** Write a polynomial to represent the profit generated by the product.
- **39.** Find the profit from sales of 1850 units.
- **40.** Simplify \((c^2 - 6cd - 2d^2) + (7c^2 - cd + 8d^2) - (-c^2 + 5cd - d^2)\).
- **41.** Find the product of \(x^2 + 6x - 5\) and \(-3x + 2\).

Simplify.

42. \((4x^2 - 3y^2 + 5xy) - (8xy + 3y^2)\)
43. \((10x^2 - 3xy + 4y^2) - (3x^2 + 5xy)\)
44. \(\frac{3}{4}x^2(8x + 12y - 16xy^2)\)
45. \(\frac{1}{2}a^3(4a - 6b + 8ab^4)\)
46. \(d^{-3}(d^5 - 2d^3 + d^{-1})\)
47. \(x^{-3}y^2(yx^4 + y^{-1}x^3 + y^{-2}x^2)\)
48. \((a^3 - b)(a^3 + b)\)
49. \((m^2 - 5)(2m^2 + 3)\)
50. \((x - 3y)^2\)
51. \((1 + 4c)^2\)

52. **GENETICS** Suppose \(R\) and \(W\) represent two genes that a plant can inherit from its parents. The terms of the expansion of \((R + W)^2\) represent the possible pairings of the genes in the offspring. Write \((R + W)^2\) as a polynomial.

53. **OPEN ENDED** Write a polynomial of degree 5 that has three terms.

54. **Which One Doesn’t Belong?** Identify the expression that does not belong with the other three. Explain your reasoning.

\[
\begin{align*}
3xy + 6x^2 & \quad & \frac{5}{x^2} & \quad & x + 5 & \quad & 5b + 11c - 9a^2
\end{align*}
\]

55. **CHALLENGE** What is the degree of the product of a polynomial of degree 8 and a polynomial of degree 6? Include an example to support your answer.

56. **Writing in Math** Use the information about tuition increases to explain how polynomials can be applied to financial situations. Include an explanation of how a polynomial can be applied to a situation with a fixed percent rate of increase and an explanation of how to use an expression and the 6% rate of increase to estimate Shenequa’s tuition in the fourth year.
57. **ACT/SAT** Which polynomial has degree 3?
   - A. \(x^3 + x^2 - 2x^4\)
   - B. \(-2x^2 - 3x + 4\)
   - C. \(x^2 + x + 12^3\)
   - D. \(1 + x + x^3\)

58. **REVIEW**
   \((-4x^2 + 2x + 3) - 3(2x^2 - 5x + 1) =
   - F. \(2x^2\)
   - G. \(-10x^2\)
   - H. \(-10x^2 + 17x\)
   - J. \(2x^2 + 17x\)

---

**Spiral Review**

Simplify. Assume that no variable equals 0. (Lesson 6-1)

59. \((-4d^2)^3\)
60. \(5rt^2(2rt)^2\)
61. \(\frac{x^2y^2z^4}{xy^3z^2}\)
62. \(\frac{(3ab^2)^2}{(6a^2b)}\)

Graph each inequality. (Lesson 5-8)

63. \(y > x^2 - 4x + 6\)
64. \(y \leq -x^2 + 6x - 3\)
65. \(y < x^2 - 2x\)

Determine whether each function has a maximum or a minimum value. Then find the maximum or minimum value of each function. (Lesson 5-1)

66. \(f(x) = x^2 - 8x + 3\)
67. \(f(x) = -3x^2 - 18x + 5\)
68. \(f(x) = -7 + 4x^2\)

Use matrices \(A, B, C,\) and \(D\) to find the following. (Lesson 4-2)

\[A = \begin{bmatrix} -4 & 4 \\ 2 & -3 \\ 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 0 \\ 4 & 1 \\ 6 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} -4 & -5 \\ -3 & 1 \\ 2 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ -3 & 4 \end{bmatrix}\]

69. \(A + D\)
70. \(B - C\)
71. \(3B - 2A\)

Write an equation in slope-intercept form for each graph. (Lesson 2-4)

72. \(y = \frac{4}{3}x - 3\)
73. \(y = \frac{3}{4}x - 2\)

74. In 1990, 2,573,225 people attended St. Louis Cardinals home games. In 2004, the attendance was 3,048,427. What was the average annual rate of increase in attendance?

**GET READY for the Next Lesson**

**PREREQUISITE SKILL** Simplify. Assume that no variable equals 0. (Lesson 6-1)

75. \(\frac{x^3}{x}\)
76. \(\frac{4y^5}{2y^2}\)
77. \(\frac{x^2y^3}{xy}\)
78. \(\frac{9a^3b}{3ab}\)
Arianna needed $140x^2 + 60x$ square inches of paper to make a book jacket $10x$ inches tall. In figuring the area she needed, she allowed for a front and back flap. If the spine of the book jacket is $2x$ inches, and the front and back of the book jacket are $6x$ inches, how wide are the front and back flaps? You can use a quotient of polynomials to help you find the answer.

**Use Long Division** In Lesson 6-1, you learned to divide monomials. You can divide a polynomial by a monomial by using those same skills.

**EXAMPLE** Divide a Polynomial by a Monomial

Simplify $\frac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy}$.

\[
\frac{4x^3y^2 + 8xy^2 - 12x^2y^3}{4xy} = \frac{4x^3y^2}{4xy} + \frac{8xy^2}{4xy} - \frac{12x^2y^3}{4xy}
\]

Sum of quotients

\[
= \frac{4}{4} \cdot x^2 - \frac{1}{4}y^2 - 1 + \frac{8}{4} \cdot x^1 - \frac{1}{4}y^2 - 1
\]

Divide.

\[
= \frac{12}{4} \cdot x^2 - \frac{1}{4}y^3 - 1
\]

\[
= x^2y + 2y - 3xy^2
\]

You can use a process similar to long division to divide a polynomial by a polynomial with more than one term. The process is known as the division algorithm. When doing the division, remember that you can only add or subtract like terms.
Which expression is equal to \((t^2 + 3t - 9)(5 - t)^{-1}\)?

A \( t + 8 - \frac{31}{5 - t} \)  
B \(-t - 8\)  
C \(-t - 8 + \frac{31}{5 - t}\)  
D \(-t - 8 - \frac{31}{5 - t}\)

Read the Test Item
Since the second factor has an exponent of \(-1\), this is a division problem. 
\((t^2 + 3t - 9)(5 - t)^{-1} = \frac{t^2 + 3t - 9}{5 - t}\)

Solve the Test Item
\(-t + 5) \quad \frac{t^2 + 3t - 9}{5 - t} \quad \frac{-t^2 + 5t}{8t - 9} \quad \frac{-8t + 40}{31}\) 
For ease in dividing, rewrite \(5 - t\) as \(-t + 5\). 
\(-t(-t + 5) = t^2 - 5t\) 
\(3t - (-5t) = 8t\) 
\(-8(-t + 5) = 8t - 40\) 
Subtract. \(-9 - (-40) = 31\) 
The quotient is \(-t - 8\), and the remainder is 31. Therefore, 
\((t^2 + 3t - 9)(5 - t)^{-1} = \frac{-t - 8 + \frac{31}{5 - t}}{5 - t}\). The answer is C.
Use Synthetic Division

Synthetic division is a simpler process for dividing a polynomial by a binomial. Suppose you want to divide \(5x^3 - 13x^2 + 10x - 8\) by \(x - 2\) using long division.

Compare the coefficients in this division with those in Example 4.

**EXAMPLE**

**Synthetic Division**

4 Use synthetic division to find \((5x^3 - 13x^2 + 10x - 8) \div (x - 2)\).

**Step 1** Write the terms of the dividend so that the degrees of the terms are in descending order. Then write just the coefficients as shown at the right.

\[
\begin{array}{ccc}
5 & -13 & 10 & -8 \\
\hline 
5 & -13 & 10 & -8 \\
\end{array}
\]

**Step 2** Write the constant \(r\) of the divisor \(x - r\) to the left. In this case, \(r = 2\).

Bring the first coefficient, 5, down.

**Step 3** Multiply the first coefficient by \(r\): \(2 \cdot 5 = 10\). Write the product under the second coefficient. Then add the product and the second coefficient: \(-13 + 10 = -3\).

**Step 4** Multiply the sum, \(-3\), by \(r\): \(2(-3) = -6\). Write the product under the next coefficient and add: \(10 + (-6) = 4\).

**Step 5** Multiply the sum, 4, by \(r\): \(2 \cdot 4 = 8\). Write the product under the next coefficient and add: \(-8 + 8 = 0\). The remainder is 0.

The numbers along the bottom row are the coefficients of the quotient. Start with the power of \(x\) that is one less than the degree of the dividend. Thus, the quotient is \(5x^2 - 3x + 4\).

**CHECK Your Progress**

Use synthetic division to find each quotient.

4A. \((2x^3 + 3x^2 - 4x + 15) \div (x + 3)\) 4B. \((3x^3 - 8x^2 + 11x - 14) \div (x - 2)\)

To use synthetic division, the divisor must be of the form \(x - r\). If the coefficient of \(x\) in a divisor is not 1, you can rewrite the division expression so that you can use synthetic division.

**EXAMPLE**

**Divisor with First Coefficient Other than 1**

5 Use synthetic division to find \((8x^4 - 4x^2 + x + 4) \div (2x + 1)\).

Use division to rewrite the divisor so it has a first coefficient of 1.

\[
\frac{8x^4 - 4x^2 + x + 4}{2x + 1} = \frac{(8x^4 - 4x^2 + x + 4) \div 2}{(2x + 1) \div 2} = \frac{4x^4 - 2x^2 + \frac{1}{2}x + 2}{x + \frac{1}{2}}
\]
Since the numerator does not have an $x^3$-term, use a coefficient of 0 for $x^3$.

$x - r = x + \frac{1}{2}$, so $r = -\frac{1}{2}$.

\[ \begin{array}{c|cccc}
-\frac{1}{2} & 4 & 0 & -2 & 1 \\
\hline
\ & 8 & 4 & -1 & 1 \\
\end{array} \]

The result is $4x^3 - 2x^2 - x + 1 + \frac{3}{x + \frac{1}{2}}$. Now simplify the fraction.

\[ \frac{\frac{3}{2}}{x + \frac{1}{2}} = \frac{3}{2} \div \left(x + \frac{1}{2}\right) \quad \text{Rewrite as a division expression.} \]

\[ = \frac{3}{2} \div \frac{2x + 1}{2} \quad \text{Multiply by the reciprocal.} \]

\[ = \frac{3}{2} \cdot \frac{2}{2x + 1} \quad \text{The solution is } 4x^3 - 2x^2 - x + 1 + \frac{3}{2x + 1}. \]

**CHECK** Divide using long division.

\[
\begin{array}{c|cccccccc}
& 4x^3 & -2x^2 & -x & 1 \\
\hline
2x + 1) & 8x^4 & +0x^3 & -4x^2 & +x & +4 \\
\hline
& -8x^4 & +4x^3 & \\
\hline
& -4x^3 & -4x^2 \\
\hline
& (-) & -4x^3 & -2x^2 \\
\hline
& & -2x^2 & -x \\
\hline
& & (-) & -2x^2 & -x \\
\hline
& & & & 2x & +4 \\
\hline
& & & & (-) & 2x & +1 \\
\hline
& & & & & \frac{3}{3} \\
\end{array}
\]

The result is $4x^3 - 2x^2 - x + 1 + \frac{3}{2x + 1}$. ✓

**CHECK Your Progress** Use synthetic division to find each quotient.

5A. $(3x^4 - 5x^3 + x^2 + 7x) \div (3x + 1)$  
5B. $(8y^5 - 2y^4 - 16y^2 + 4) \div (4y - 1)$

**Example 1** (p. 325)

Simplify.

1. \[
\frac{6xy^2 - 3xy + 2x^2y}{xy}
\]

2. \[
(5ab^2 - 4ab + 7a^2b)(ab)^{-1}
\]

3. **BAKING** The number of cookies produced in a factory each day can be estimated by $C(w) = -w^2 + 16w + 1000$, where $w$ is the number of workers and $C$ is the number of cookies produced. Divide to find the average number of cookies produced per worker.

Simplify.

4. \[
(x^2 - 10x - 24) \div (x + 2)
\]

5. \[
(3a^4 - 6a^3 - 2a^2 + a - 6) \div (a + 1)
\]

6. \[
(z^3 - 3z^2 - 20) \div (z - 2)
\]

7. \[
(x^3 + y^3) \div (x + y)
\]

8. \[
\frac{x^3 + 13x^2 - 12x - 8}{x + 2}
\]

9. \[
(b^4 - 2b^3 + b^2 - 3b + 4)(b - 2)^{-1}
\]
10. **STANDARDIZED TEST PRACTICE** Which expression is equal to $(x^2 - 4x + 6)(x - 3)^{-1}$?

   A. $x - 1$  
   B. $x - 1 + \frac{3}{x - 3}$  
   C. $x - 1 - \frac{3}{x - 3}$  
   D. $-x + 1 - \frac{3}{x - 3}$

**Simplify.**

11. $(12y^2 + 36y + 15) \div (6y + 3)$

12. $\frac{9b^2 + 9b - 10}{3b - 2}$

**Exercises**

<table>
<thead>
<tr>
<th>HOMEWORK HELP</th>
<th>For Exercises</th>
<th>See Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>13. $\frac{9a^3b^2 - 18a^2b^3}{3a^2b}$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>15. $(28c^3d - 42cd^2 + 56cd^3) \div (14cd)$</td>
<td>3, 4</td>
<td></td>
</tr>
<tr>
<td>17. $(x^3 - 4x^2) \div (x - 4)$</td>
<td>2, 3, 5</td>
<td></td>
</tr>
<tr>
<td>19. $(b^3 + 8b^2 - 20b) \div (b - 2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>21. $\frac{y^3 + 3y^2 - 5y - 4}{y + 4}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>23. $(t^5 - 3t^2 - 20)(t - 2)^{-1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>25. $(2c^3 - 3c^2 + 3c - 4) \div (c - 2)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>27. $\frac{x^5 - 7x^3 + x + 1}{x + 3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>29. $\frac{4x^3 + 5x^2 - 3x + 1}{4x + 1}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>31. $(6t^3 + 5t^2 + 9) \div (2t + 3)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>33. $\frac{2x^4 + 3x^3 - 2x^2 - 3x - 6}{2x + 3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>35. <strong>ENTERTAINMENT</strong> A magician gives these instructions to a volunteer.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; Choose a number and multiply it by 4.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; Then add the sum of your number and 15 to the product you found.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; Now divide by the sum of your number and 3.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>&amp; What number will the volunteer always have at the end? Explain.</td>
<td></td>
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</tr>
</tbody>
</table>

**BUSINESS** For Exercises 36 and 37, use the following information.

The number of sports magazines sold can be estimated by $n = \frac{3500a^2}{a^2 + 100}$, where $a$ is the amount of money spent on advertising in hundreds of dollars and $n$ is the number of subscriptions sold.

36. Perform the division indicated by $\frac{3500a^2}{a^2 + 100}$.

37. About how many subscriptions will be sold if $1500$ is spent on advertising?

**PHYSICS** For Exercises 38–40, suppose an object moves in a straight line so that, after $t$ seconds, it is $t^3 + t^2 + 6t$ feet from its starting point.

38. Find the distance the object travels between the times $t = 2$ and $t = x$, where $x > 2$.

39. How much time elapses between $t = 2$ and $t = x$?

40. Find a simplified expression for the average speed of the object between times $t = 2$ and $t = x$.  

---

**Real-World Career**

**Cost Analyst**

Cost analysts study and write reports about the factors involved in the cost of production.

For more information, go to algebra2.com.
41. **OPEN ENDED** Write a quotient of two polynomials such that the remainder is 5.

42. **REASONING** Review any of the division problems in this lesson. What is the relationship between the degrees of the dividend, the divisor, and the quotient?

43. **FIND THE ERROR** Shelly and Jorge are dividing $x^3 - 2x^2 + x - 3$ by $x - 4$. Who is correct? Explain your reasoning.

Shelly

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<th>1</th>
<th>-3</th>
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<td>100</td>
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<td>1</td>
<td>-6</td>
<td>25</td>
<td>-103</td>
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Jorge

<table>
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<tr>
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<th>1</th>
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<th>-3</th>
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</tr>
<tr>
<td>1</td>
<td>2</td>
<td>9</td>
<td>33</td>
<td></td>
</tr>
</tbody>
</table>

44. **CHALLENGE** Suppose the result of dividing one polynomial by another is $r^2 - 6r + 9 - \frac{1}{r - 3}$. What two polynomials might have been divided?

45. **Writing in Math** Use the information on page 325 to explain how you can use division of polynomials in manufacturing. Include the dimensions of the piece of paper that the publisher needs, the formula from geometry that applies to this situation, and an explanation of how to use division of polynomials to find the width of the flap.

46. **ACT/SAT** What is the remainder when $x^3 - 7x + 5$ is divided by $x + 3$?

A. $-11$  
B. $-1$  
C. $1$  
D. $11$

47. **REVIEW** If $i = \sqrt{-1}$, then $5i(7i) =$

F. $70$  
H. $-35$  
G. $35$  
J. $-70$

48. $(2x^2 - 3x + 5) - (3x^2 + x - 9)$

49. $y^2z(y^2z^3 - yz^2 + 3)$

50. $(y + 5)(y - 3)$

51. $(a - b)^2$

52. **ASTRONOMY** Earth is an average of $1.5 \times 10^{11}$ meters from the Sun. Light travels at $3 \times 10^8$ meters per second. About how long does it take sunlight to reach Earth? (Lesson 6-1)

53. $f(-2)$  
54. $f(2)$  
55. $f(2a)$  
56. $f(a + 1)$

**PREREQUISITE SKILL** Given $f(x) = x^2 - 5x + 6$, find each value. (Lesson 2-1)
Polynomial Functions

A cross section of a honeycomb has a pattern with one hexagon surrounded by six more hexagons. Surrounding these is a third ring of 12 hexagons, and so on. The total number of hexagons in a honeycomb can be modeled by the function \( f(r) = 3r^2 - 3r + 1 \), where \( r \) is the number of rings and \( f(r) \) is the number of hexagons.

**Polynomial Functions** The expression \( 3r^2 - 3r + 1 \) is a polynomial in one variable since it only contains one variable, \( r \).

### KEY CONCEPT

**Polynomial in One Variable**

<table>
<thead>
<tr>
<th>Terms</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Words</strong></td>
<td>A polynomial of degree ( n ) in one variable ( x ) is an expression of the form ( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 ), where the coefficients ( a_n, a_{n-1}, \ldots, a_2, a_1, a_0 ) represent real numbers, ( a_n ) is not zero, and ( n ) represents a nonnegative integer.</td>
</tr>
<tr>
<td><strong>Example</strong></td>
<td>( 3x^5 + 2x^4 - 5x^3 + x^2 + 1 ) ( n = 5, a_5 = 3, a_4 = 2, a_3 = -5, a_2 = 1, a_1 = 0, ) and ( a_0 = 1 )</td>
</tr>
</tbody>
</table>

The degree of a polynomial in one variable is the greatest exponent of its variable. The **leading coefficient** is the coefficient of the term with the highest degree.

<table>
<thead>
<tr>
<th>Polynomial</th>
<th>Expression</th>
<th>Degree</th>
<th>Leading Coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>Constant</td>
<td>9</td>
<td>0</td>
<td>9</td>
</tr>
<tr>
<td>Linear</td>
<td>( x - 2 )</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Quadratic</td>
<td>( 3x^2 + 4x - 5 )</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Cubic</td>
<td>( 4x^3 - 6 )</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>General</td>
<td>( a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 )</td>
<td>( n )</td>
<td>( a_n )</td>
</tr>
</tbody>
</table>

### EXAMPLE

**Find Degrees and Leading Coefficients**

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

**a.** \( 7x^4 + 5x^2 + x - 9 \)

This is a polynomial in one variable.

The degree is 4, and the leading coefficient is 7.

(continued on the next page)
b. $8x^2 + 3xy - 2y^2$

This is not a polynomial in one variable. It contains two variables, $x$ and $y$.

---

A polynomial equation used to represent a function is called a polynomial function. For example, the equation $f(x) = 4x^2 - 5x + 2$ is a quadratic polynomial function, and the equation $p(x) = 2x^3 + 4x^2 - 5x + 7$ is a cubic polynomial function. Other polynomial functions can be defined by the following general rule.

**Definition of a Polynomial Function**

**Words**
A polynomial function of degree $n$ is a continuous function that can be described by an equation of the form $P(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$, where the coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ represent real numbers, $a_n$ is not zero, and $n$ represents a nonnegative integer.

**Example**

$f(x) = 4x^2 - 3x + 2$

$n = 2, a_2 = 4, a_1 = -3, a_0 = 2$

If you know an element in the domain of any polynomial function, you can find the corresponding value in the range. Recall that $f(3)$ can be found by evaluating the function for $x = 3$.

---

**NATURE** Refer to the application at the beginning of the lesson.

**a.** Show that the polynomial function $f(r) = 3r^2 - 3r + 1$ gives the total number of hexagons when $r = 1, 2,$ and $3$.

Find the values of $f(1), f(2),$ and $f(3)$.

$f(r) = 3r^2 - 3r + 1$

$f(1) = 3(1)^2 - 3(1) + 1 = 1 = 1$

$f(2) = 3(2)^2 - 3(2) + 1 = 12 - 6 + 1 = 7$

$f(3) = 3(3)^2 - 3(3) + 1 = 27 - 9 + 1 = 19$

You know the numbers of hexagons in the first three rings are 1, 6, and 12. So, the total number of hexagons with one ring is 1, two rings is 6 + 1 or 7, and three rings is 12 + 6 + 1 or 19. These match the functional values for $r = 1, 2,$ and $3$, respectively. That is 1, 7, and 19 are the range values corresponding to the domain values of 1, 2, and 3.

**b.** Find the total number of hexagons in a honeycomb with 12 rings.

$f(r) = 3r^2 - 3r + 1$  
Original function

$f(12) = 3(12)^2 - 3(12) + 1$  
Replace $r$ with 12.

$= 432 - 36 + 1$ or 397  
Simplify.

---

2A. Show that $f(r)$ gives the total number of hexagons when $r = 4$.

2B. Find the total number of hexagons in a honeycomb with 20 rings.
You can also evaluate functions for variables and algebraic expressions.

**Function Values of Variables**

Find \(q(a + 1) - 2q(a)\) if \(q(x) = x^2 + 3x + 4\).

To evaluate \(q(a + 1)\), replace \(x\) in \(q(x)\) with \(a + 1\).

\[
q(x) = x^2 + 3x + 4 \quad \text{Original function}
\]

\[
q(a + 1) = (a + 1)^2 + 3(a + 1) + 4 \quad \text{Replace } x \text{ with } a + 1.
\]

\[
= a^2 + 2a + 1 + 3a + 3 + 4 \quad \text{Simplify } (a + 1)^2 \text{ and } 3(a + 1).
\]

\[
= a^2 + 5a + 8 \quad \text{Simplify}.
\]

To evaluate \(2q(a)\), replace \(x\) with \(a\) in \(q(x)\), then multiply the expression by 2.

\[
q(x) = x^2 + 3x + 4 \quad \text{Original function}
\]

\[
2q(a) = 2(a^2 + 3a + 4) \quad \text{Replace } x \text{ with } a.
\]

\[
= 2a^2 + 6a + 8 \quad \text{Distributive Property}
\]

Now evaluate \(q(a + 1) - 2q(a)\).

\[
q(a + 1) - 2q(a) = a^2 + 5a + 8 - (2a^2 + 6a + 8) \quad \text{Replace } q(a + 1) \text{ and } 2q(a).
\]

\[
= a^2 + 5a + 8 - 2a^2 - 6a - 8
\]

\[
= -a^2 - a \quad \text{Simplify}.
\]

**Check Your Progress**

3A. Find \(f(b^2)\) if \(f(x) = 2x^2 + 3x - 1\).

3B. Find \(2g(c + 2) + 3g(2c)\) if \(g(x) = x^2 - 4\).

**Graphs of Polynomial Functions** The general shapes of the graphs of several polynomial functions are shown below. These graphs show the maximum number of times the graph of each type of polynomial may intersect the \(x\)-axis. Recall that the \(x\)-coordinate of the point at which the graph intersects the \(x\)-axis is called a zero of a function. How does the degree compare to the maximum number of real zeros?

**Math Nihle**

Extra Examples at algebra2.com
The **end behavior** is the behavior of the graph as \( x \) approaches positive infinity \((+\infty)\) or negative infinity \((-\infty)\). This is represented as \( x \to +\infty \) and \( x \to -\infty \), respectively. \( x \to +\infty \) is read **\( x \) approaches positive infinity**. Notice the shapes of the graphs for even-degree polynomial functions and odd-degree polynomial functions. The degree and leading coefficient of a polynomial function determine the graph’s end behavior.

### Concept Summary

#### End Behavior of a Polynomial Function

<table>
<thead>
<tr>
<th>Degree</th>
<th>Leading Coefficient</th>
<th>End Behavior</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>positive</td>
<td>( f(x) \to + ) as ( x \to -\infty ) ( f(x) \to + ) as ( x \to +\infty )</td>
</tr>
<tr>
<td>Odd</td>
<td>positive</td>
<td>( f(x) \to + ) as ( x \to +\infty ) ( f(x) \to - ) as ( x \to -\infty )</td>
</tr>
<tr>
<td>Even</td>
<td>negative</td>
<td>( f(x) \to + ) as ( x \to +\infty ) ( f(x) \to - ) as ( x \to -\infty )</td>
</tr>
<tr>
<td>Odd</td>
<td>negative</td>
<td>( f(x) \to - ) as ( x \to -\infty ) ( f(x) \to - ) as ( x \to +\infty )</td>
</tr>
</tbody>
</table>

#### Domain
- All real numbers

#### Range
- All real numbers

For any polynomial function, the domain is all real numbers. For any polynomial function of odd degree, the range is all real numbers. For polynomial functions of even degree, the range is all real numbers greater than or equal to some number or all real numbers less than or equal to some number; it is never all real numbers.

The graph of an even-degree function may or may not intersect the \( x \)-axis. If it intersects the \( x \)-axis in two places, the function has two real zeros. If it does not intersect the \( x \)-axis, the roots of the related equation are imaginary and cannot be determined from the graph. If the graph is tangent to the \( x \)-axis, as shown above, there are two zeros that are the same number. The graph of an odd-degree function always crosses the \( x \)-axis at least once, and thus the function always has at least one real zero.

### Example

**Graphs of Polynomial Functions**

For each graph,
- **describe the end behavior,**
- **determine whether it represents an odd-degree or an even-degree polynomial function,** and
- **state the number of real zeros.**

### a. [Graph]

The same is true for an even-degree function. One exception is when the graph of \( f(x) \) touches the \( x \)-axis.

### b. [Graph]
a. \( f(x) \to -\infty \) as \( x \to +\infty \). \( f(x) \to -\infty \) as \( x \to -\infty \).

- It is an even-degree polynomial function.
- The graph intersects the x-axis at two points, so the function has two real zeros.

b. \( f(x) \to +\infty \) as \( x \to +\infty \). \( f(x) \to +\infty \) as \( x \to -\infty \).

- It is an even-degree polynomial function.
- This graph does not intersect the x-axis, so the function has no real zeros.

### Example 1
(pp. 331–332)

State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

1. \( 5x^6 - 8x^2 \)
2. \( 2b + 4b^3 - 3b^5 - 7 \)

### Example 2
(p. 332)

Find \( p(3) \) and \( p(-1) \) for each function.

3. \( p(x) = -x^3 + x^2 - x \)
4. \( p(x) = x^4 - 3x^3 + 2x^2 - 5x + 1 \)

5. **BIOLOGY** The intensity of light emitted by a firefly can be determined by \( L(t) = 10 + 0.3t + 0.4t^2 - 0.01t^3 \), where \( t \) is temperature in degrees Celsius and \( L(t) \) is light intensity in lumens. If the temperature is 30°C, find the light intensity.

### Example 3
(p. 333)

If \( p(x) = 2x^3 + 6x - 12 \) and \( q(x) = 5x^2 + 4 \), find each value.

6. \( p(a^3) \)
7. \( 5[q(2a)] \)
8. \( 3p(a) - q(a + 1) \)

### Example 4
(pp. 334–335)

For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.
State the degree and leading coefficient of each polynomial in one variable. If it is not a polynomial in one variable, explain why.

12. \(7 - x\)  
13. \((a + 1)(a^2 - 4)\)  
14. \(a^2 + 2ab + b^2\)  
15. \(c^2 + c - \frac{1}{c}\)  
16. \(6x^4 + 3x^2 + 4x - 8\)  
17. \(7 + 3x^2 - 5x^3 + 6x^2 - 2x\)

Find \(p(4)\) and \(p(-2)\) for each function.

18. \(p(x) = 2 - x\)  
19. \(p(x) = x^2 - 3x + 8\)  
20. \(p(x) = 2x^3 - x^2 + 5x - 7\)  
21. \(p(x) = x^5 - x^2\)

If \(p(x) = 3x^2 - 2x + 5\) and \(r(x) = x^3 + x + 1\), find each value.

22. \(r(3a)\)  
23. \(4p(a)\)  
24. \(p(a^2)\)  
25. \(p(2a^3)\)  
26. \(r(x + 1)\)  
27. \(p(x^2 + 3)\)

For each graph,

a. describe the end behavior,

b. determine whether it represents an odd-degree or an even-degree polynomial function, and

c. state the number of real zeros.

34. **ENERGY**  The power generated by a windmill is a function of the speed of the wind. The approximate power is given by the function \(P(s) = \frac{s^3}{1000}\), where \(s\) represents the speed of the wind in kilometers per hour. Find the units of power \(P(s)\) generated by a windmill when the wind speed is 18 kilometers per hour.

35. **PHYSICS**  For a moving object with mass \(m\) in kilograms, the kinetic energy \(KE\) in joules is given by the function \(KE(v) = \frac{1}{2}mv^2\), where \(v\) represents the speed of the object in meters per second. Find the kinetic energy of an all-terrain vehicle with a mass of 171 kilograms moving at a speed of 11 meters/second.

Find \(p(4)\) and \(p(-2)\) for each function.

36. \(p(x) = x^4 - 7x^3 + 8x - 6\)  
37. \(p(x) = 7x^2 - 9x + 10\)  
38. \(p(x) = \frac{1}{2}x^3 - 2x^2 + 4\)  
39. \(p(x) = \frac{1}{8}x^3 - \frac{1}{4}x^2 - \frac{1}{2}x + 5\)
If \( p(x) = 3x^2 - 2x + 5 \) and \( r(x) = x^3 + x + 1 \), find each value.

40. \( 2[p(x + 4)] \)  
41. \( r(x + 1) - r(x^2) \)  
42. \( 3[p(x^2 - 1)] + 4p(x) \)

**THEATER** For Exercises 43–45, use the graph that models the attendance at Broadway plays (in millions) from 1985–2005.

43. Is the graph an odd-degree or even-degree function?
44. Discuss the end behavior.
45. Do you think attendance at Broadway plays will increase or decrease after 2005? Explain your reasoning.

**PATTERNS** For Exercises 46–48, use the diagrams below that show the maximum number of regions formed by connecting points on a circle.

<table>
<thead>
<tr>
<th>Points</th>
<th>Regions</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

46. The number of regions formed by connecting \( n \) points of a circle can be described by the function \( f(n) = \frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24) \). What is the degree of this polynomial function?
47. Find the number of regions formed by connecting 5 points of a circle. Draw a diagram to verify your solution.
48. How many points would you have to connect to form 99 regions?

49. **REASONING** Explain why a constant polynomial such as \( f(x) = 4 \) has degree 0 and a linear polynomial such as \( f(x) = x + 5 \) has degree 1.

50. **OPEN ENDED** Sketch the graph of an odd-degree polynomial function with a negative leading coefficient and three real roots.

51. **REASONING** Determine whether the following statement is always, sometimes or never true. Explain.

A polynomial function that has four real roots is a fourth-degree polynomial.

**CHALLENGE** For Exercises 52–55, use the following information.

The graph of the polynomial function \( f(x) = ax(x - 4)(x + 1) \) goes through the point at (5, 15).

52. Find the value of \( a \).
53. For what value(s) of \( x \) will \( f(x) = 0 \)?
54. Simplify and rewrite the function as a cubic function.
55. Sketch the graph of the function.

56. **Writing in Math** Use the information on page 331 to explain where polynomial functions are found in nature. Include an explanation of how you could use the equation to find the number of hexagons in the tenth ring and any other examples of patterns found in nature that might be modeled by a polynomial equation.
57. **ACT/SAT** The figure at the right shows the graph of a polynomial function \( f(x) \). Which of the following could be the degree of \( f(x) \)?

- A 2
- B 3

58. **REVIEW** Which polynomial represents \((4x^2 + 5x - 3)(2x - 7)\)?

- F \( 8x^3 - 18x^2 - 41x - 21 \)
- G \( 8x^3 + 18x^2 + 29x - 21 \)
- H \( 8x^3 - 18x^2 - 41x + 21 \)
- J \( 8x^3 + 18x^2 - 29x + 21 \)

---

**Spiral Review**

Simplify. (Lesson 6-3)

59. \( (t^3 - 3t + 2) \div (t + 2) \)

60. \( (y^2 + 4y + 3)(y + 1)^{-1} \)

61. \( \frac{x^3 - 3x^2 + 2x - 6}{x - 3} \)

62. \( \frac{3x^4 + x^3 - 8x^2 + 10x - 3}{3x - 2} \)

63. **BUSINESS** Ms. Schifflet is writing a computer program to find the salaries of her employees after their annual raise. The percent of increase is represented by \( p \). Marty’s salary is $23,450 now. Write a polynomial to represent Marty’s salary in one year and another to represent Marty’s salary after three years. Assume that the rate of increase will be the same for each of the three years. (Lesson 6-2)

Solve each equation by completing the square. (Lesson 5-5)

64. \( x^2 - 8x - 2 = 0 \)

65. \( x^2 + \frac{1}{3}x - \frac{35}{36} = 0 \)

Write an absolute value inequality for each graph. (Lesson 1-6)

66. \[ -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

67. \[ -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

68. \[ -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \]

69. \[ -5 \quad -4 \quad -3 \quad -2 \quad -1 \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \]

Name the property illustrated by each statement. (Lesson 1-3)

70. If \( 3x = 4y \) and \( 4y = 15z \), then \( 3x = 15z \).

71. \( 5y(4a - 6b) = 20ay - 30by \)

72. \( 2 + (3 + x) = (2 + 3) + x \)

---

**GET READY for the Next Lesson**

**PREREQUISITE SKILL** Graph each equation by making a table of values. (Lesson 5-1)

73. \( y = x^2 + 4 \)

74. \( y = -x^2 + 6x - 5 \)

75. \( y = \frac{1}{2}x^2 + 2x - 6 \)
The percent of the United States population that was foreign-born since 1900 can be modeled by \( P(t) = 0.00006t^3 - 0.007t^2 + 0.05t + 14 \), where \( t = 0 \) in 1900. Notice that the graph is decreasing from \( t = 5 \) to \( t = 75 \) and then it begins to increase. The points at \( t = 5 \) and \( t = 75 \) are turning points in the graph.

**Graph Polynomial Functions** To graph a polynomial function, make a table of values to find several points and then connect them to make a smooth continuous curve. Knowing the end behavior of the graph will assist you in completing the sketch of the graph.

**EXAMPLE**

### Graph a Polynomial Function

Graph \( f(x) = x^4 + x^3 - 4x^2 - 4x \) by making a table of values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2.5</td>
<td>( \approx 8.4 )</td>
</tr>
<tr>
<td>-2.0</td>
<td>0.0</td>
</tr>
<tr>
<td>-1.5</td>
<td>( \approx -1.3 )</td>
</tr>
<tr>
<td>-1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>-0.5</td>
<td>( \approx 0.9 )</td>
</tr>
<tr>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>0.5</td>
<td>( \approx -2.8 )</td>
</tr>
<tr>
<td>1.0</td>
<td>-6.0</td>
</tr>
<tr>
<td>1.5</td>
<td>( \approx -6.6 )</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0</td>
</tr>
</tbody>
</table>

This is an even-degree polynomial with a positive leading coefficient, so \( f(x) \to +\infty \) as \( x \to +\infty \), and \( f(x) \to +\infty \) as \( x \to -\infty \). Notice that the graph intersects the \( x \)-axis at four points, indicating there are four real zeros of this function.

**CHECK Your Progress**

1. Graph \( f(x) = x^4 - x^3 - x^2 + x \) by making a table of values.
In Example 1, the zeros occur at integral values that can be seen in the table used to plot the function. Notice that the values of the function before and after each zero are different in sign. In general, because it is a continuous function, the graph of a polynomial function will cross the x-axis somewhere between pairs of x-values at which the corresponding f(x)-values change signs. Since zeros of the function are located at x-intercepts, there is a zero between each pair of these x-values. This property for locating zeros is called the **Location Principle**.

---

**KEY CONCEPT**

**Location Principle**

**Words**
Suppose \( y = f(x) \) represents a polynomial function and \( a \) and \( b \) are two numbers such that \( f(a) < 0 \) and \( f(b) > 0 \). Then the function has at least one real zero between \( a \) and \( b \).

**Model**

The changes in sign indicate that there are zeros between \( x = -1 \) and \( x = 0 \), between \( x = 1 \) and \( x = 2 \), and between \( x = 4 \) and \( x = 5 \).

---

**EXAMPLE**

**Locate Zeros of a Function**

1. Determine consecutive integer values of \( x \) between which each real zero of the function \( f(x) = x^3 - 5x^2 + 3x + 2 \) is located. Then draw the graph.

Make a table of values. Since \( f(x) \) is a third-degree polynomial function, it will have either 1, 2, or 3 real zeros. Look at the values of \( f(x) \) to locate the zeros. Then use the points to sketch a graph of the function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-32</td>
</tr>
<tr>
<td>-1</td>
<td>-7</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>-7</td>
</tr>
<tr>
<td>4</td>
<td>-2</td>
</tr>
<tr>
<td>5</td>
<td>17</td>
</tr>
</tbody>
</table>

The changes in sign indicate that there are zeros between \( x = -1 \) and \( x = 0 \), between \( x = 1 \) and \( x = 2 \), and between \( x = 4 \) and \( x = 5 \).

---

**CHECK Your Progress**

2. Determine consecutive integer values of \( x \) between which each real zero of the function \( f(x) = x^3 + 4x^2 - 6x - 7 \) is located. Then draw the graph.

---

**Maximum and Minimum Points**

The graph at the right shows the shape of a general third-degree polynomial function.

Point \( A \) on the graph is a **relative maximum** of the cubic function since no other nearby points have a greater \( y \)-coordinate. Likewise, point \( B \) is a **relative minimum** since no other nearby points have a lesser \( y \)-coordinate. These points are often referred to as **turning points**. The graph of a polynomial function of degree \( n \) has at most \( n - 1 \) turning points.
EXAMPLE  Maximum and Minimum Points

Graph \( f(x) = x^3 - 3x^2 + 5 \). Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.

Make a table of values and graph the equation.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-15</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
</tbody>
</table>

Look at the table of values and the graph.
- The values of \( f(x) \) change signs between \( x = -2 \) and \( x = -1 \), indicating a zero of the function.
- The value of \( f(x) \) at \( x = 0 \) is greater than the surrounding points, so it is a relative maximum.
- The value of \( f(x) \) at \( x = 2 \) is less than the surrounding points, so it is a relative minimum.

CHECK Your Progress

3. Graph \( f(x) = x^3 + 4x^2 - 3 \). Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.

The graph of a polynomial function can reveal trends in real-world data.

REAL-WORLD EXAMPLE  Graph a Polynomial Model

ENERGY  The average fuel (in gallons) consumed by individual vehicles in the United States from 1960 to 2000 is modeled by the cubic equation \( F(t) = 0.025t^3 - 1.5t^2 + 18.25t + 654 \), where \( t \) is the number of years since 1960.

a. Graph the equation.

Make a table of values for the years 1960–2000. Plot the points and connect with a smooth curve. Finding and plotting the points for every fifth year gives a good approximation of the graph.

<table>
<thead>
<tr>
<th>( t )</th>
<th>( F(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>654</td>
</tr>
<tr>
<td>5</td>
<td>710.88</td>
</tr>
<tr>
<td>10</td>
<td>711.5</td>
</tr>
<tr>
<td>15</td>
<td>674.63</td>
</tr>
<tr>
<td>20</td>
<td>619</td>
</tr>
<tr>
<td>25</td>
<td>563.38</td>
</tr>
<tr>
<td>30</td>
<td>526.5</td>
</tr>
<tr>
<td>35</td>
<td>527.13</td>
</tr>
<tr>
<td>40</td>
<td>584</td>
</tr>
</tbody>
</table>

(continued on the next page)
b. Describe the turning points of the graph and its end behavior.

There is a relative maximum between 1965 and 1970 and a relative minimum between 1990 and 1995. For the end behavior, as \( t \) increases, \( F(t) \) increases.

c. What trends in fuel consumption does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

Average fuel consumption hit a maximum point around 1970 and then started to decline until 1990. Since 1990, fuel consumption has risen and continues to rise. The trend may continue for some years, but it is unlikely that consumption will rise this quickly indefinitely. Fuel supplies will limit consumption.

4. The price of one share of stock of a company is given by the function \( f(x) = 0.001x^4 - 0.03x^3 + 0.15x^2 + 1.01x + 18.96 \), where \( x \) is the number of months since January 2006. Graph the equation. Describe the turning points of the graph and its end behavior. What trends in the stock price does the graph suggest? Is it reasonable to assume the trend will continue indefinitely?

A graphing calculator can be helpful in finding the relative maximum and relative minimum of a function.
Example 1  
(p. 339)  
Graph each polynomial function by making a table of values.  
1. \( f(x) = x^3 - x^2 - 4x + 4 \)  
2. \( f(x) = x^4 - 7x^2 + x + 5 \)

Example 2  
(p. 340)  
Determine the consecutive integer values of \( x \) between which each real zero of each function is located. Then draw the graph.  
3. \( f(x) = x^3 - x^2 + 1 \)  
4. \( f(x) = x^4 - 4x^2 + 2 \)

Example 3  
(p. 341)  
Graph each polynomial function. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur. State the domain and range for each function.  
5. \( f(x) = x^3 + 2x^2 - 3x - 5 \)  
6. \( f(x) = x^4 - 8x^2 + 10 \)

Example 4  
(pp. 341–342)  
\( \text{CABLE TV} \) For Exercises 7–10, use the following information.  
The number of cable TV systems after 1985 can be modeled by the function  
\( C(t) = -43.2t^2 + 1343t + 790 \), where \( t \) represents the number of years since 1985.  
7. Graph this equation for the years 1985 to 2005.  
8. Describe the turning points of the graph and its end behavior.  
9. What is the domain of the function? Use the graph to estimate the range.  
10. What trends in cable TV subscriptions does the graph suggest? Is it reasonable to assume that the trend will continue indefinitely?

For Exercises 11–18, complete each of the following.  
a. Graph each function by making a table of values.  
b. Determine the consecutive integer values of \( x \) between which each real zero is located.  
c. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.  
11. \( f(x) = -x^3 - 4x^2 \)  
12. \( f(x) = x^3 - 2x^2 + 6 \)  
13. \( f(x) = x^3 - 3x^2 + 2 \)  
14. \( f(x) = x^3 + 5x^2 - 9 \)  
15. \( f(x) = -3x^3 + 20x^2 - 36x + 16 \)  
16. \( f(x) = x^3 - 4x^2 + 2x - 1 \)  
17. \( f(x) = x^4 - 8 \)  
18. \( f(x) = x^4 - 10x^2 + 9 \)

\( \text{EMPLOYMENT} \) For Exercises 19–22, use the graph that models the unemployment rates from 1975–2004.  
19. In what year was the unemployment rate the highest? the lowest?  
20. Describe the turning points and the end behavior of the graph.  
21. If this graph was modeled by a polynomial equation, what is the least degree the equation could have?  
22. Do you expect the unemployment rate to increase or decrease from 2005 to 2010? Explain your reasoning.
HEALTH For Exercises 23–25, use the following information. During a regular respiratory cycle, the volume of air in liters in human lungs can be described by \( V(t) = 0.173t + 0.152t^2 - 0.035t^3 \), where \( t \) is the time in seconds.

23. Estimate the real zeros of the function by graphing.
24. About how long does a regular respiratory cycle last?
25. Estimate the time in seconds from the beginning of this respiratory cycle for the lungs to fill to their maximum volume of air.

For Exercises 26–31, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of \( x \) between which each real zero is located.

c. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.

26. \( f(x) = -x^4 + 5x^2 - 2x - 1 \)  
27. \( f(x) = -x^4 + x^3 + 8x^2 - 3 \)
28. \( f(x) = x^4 - 9x^3 + 25x^2 - 24x + 6 \)  
29. \( f(x) = 2x^4 - 4x^3 - 2x^2 + 3x - 5 \)
30. \( f(x) = x^5 + 4x^4 - x^3 - 9x^2 + 3 \)  
31. \( f(x) = x^5 - 6x^4 + 4x^3 + 17x^2 - 5x - 6 \)

CHILD DEVELOPMENT For Exercises 32 and 33, use the following information.

The average height (in inches) for boys ages 1 to 20 can be modeled by the equation \( B(x) = -0.001x^4 + 0.04x^3 - 0.56x^2 + 5.5x + 25 \), where \( x \) is the age (in years). The average height for girls ages 1 to 20 is modeled by the equation \( G(x) = -0.0002x^4 + 0.006x^3 - 0.14x^2 + 3.7x + 26 \).

32. Graph both equations by making a table of values. Use \( x = \{0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20\} \) as the domain. Round values to the nearest inch.
33. Compare the graphs. What do the graphs suggest about the growth rate for both boys and girls?

Use a graphing calculator to estimate the \( x \)-coordinates at which the maxima and minima of each function occur. Round to the nearest hundredth.

34. \( f(x) = x^3 + x^2 - 7x - 3 \)  
35. \( f(x) = -x^3 + 6x^2 - 6x - 5 \)
36. \( f(x) = -x^4 + 3x^2 - 8 \)  
37. \( f(x) = 3x^4 - 7x^3 + 4x - 5 \)

38. OPEN ENDED Sketch a graph of a function that has one relative maximum point and two relative minimum points.

CHALLENGE For Exercises 39–41, sketch a graph of each polynomial.

39. even-degree polynomial function with one relative maximum and two relative minima
40. odd-degree polynomial function with one relative maximum and one relative minimum; the leading coefficient is negative
41. odd-degree polynomial function with three relative maxima and three relative minima; the leftmost points are negative

42. REASONING Explain the Location Principle and how to use it.
43. Writing in Math Use the information about foreign-born population on page 339 to explain how graphs of polynomial functions can be used to show trends in data. Include a description of the types of data that are best modeled by polynomial functions and an explanation of how you would determine when the percent of foreign-born citizens was at its highest and when the percent was at its lowest since 1900.
44. ACT/SAT Which of the following could be the graph of \( f(x) = x^3 + x^2 - 3x \)?

A

B

C

D

45. REVIEW Mandy went shopping. She spent two-fifths of her money in the first store. She spent three-fifths of what she had left in the next store. In the last store she visited, she spent three-fourths of the money she had left. When she finished shopping, Mandy had $6. How much money in dollars did Mandy have when she started shopping?

F $16  
H $100  
G $56  
J $106

If \( p(x) = 2x^2 - 5x + 4 \) and \( r(x) = 3x^3 - x^2 - 2 \), find each value. (Lesson 6-4)

46. \( r(2a) \)

47. \( 5p(c) \)

48. \( p(2a^2) \)

49. \( r(x - 1) \)

50. \( p(x^2 + 4) \)

Simplify. (Lesson 6-3)

52. \( (4x^3 - 7x^2 + 3x - 2) \div (x - 2) \)

53. \( \frac{x^4 + 4x^3 - 4x^2 + 5x}{x - 5} \)

Simplify. (Lesson 6-2)

54. \( (3x^2 - 2xy + y^2) + (x^2 + 5xy - 4y^2) \)

55. \( (2x + 4)(7x - 1) \)

Solve each matrix equation or system of equations by using inverse matrices. (Lesson 4-8)

56. \[
\begin{bmatrix} 3 & 6 \\ 2 & -1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -3 \\ 18 \end{bmatrix}
\]

57. \[
\begin{bmatrix} 5 & -7 \\ -3 & 4 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{m} \\ \mathbf{n} \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}
\]

58. \[
3j + 2k = 8 \\
j - 7k = 18
\]

60. SPORTS Bob and Minya want to build a ramp that they can use while rollerblading. If they want the ramp to have a slope of \( \frac{4}{3} \), how tall should they make the ramp? (Lesson 2-3)

61. 18, 27  
62. 24, 84  
63. 16, 28

64. 12, 27, 48  
65. 12, 30, 54  
66. 15, 30, 65
Graphing Calculator Lab
Modeling Data Using Polynomial Functions

You can use a TI-83/84 Plus graphing calculator to model data for which the curve of best fit is a polynomial function.

**Example**

The table shows the distance a seismic wave can travel based on its distance from an earthquake’s epicenter. Draw a scatter plot and a curve of best fit that relates distance to travel time. Then determine approximately how far from the epicenter the wave will be felt 8.5 minutes after the earthquake occurs.

<table>
<thead>
<tr>
<th>Travel Time (min)</th>
<th>1</th>
<th>2</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>Distance (km)</td>
<td>400</td>
<td>800</td>
<td>2500</td>
<td>3900</td>
<td>6250</td>
<td>8400</td>
<td>10000</td>
</tr>
</tbody>
</table>

Source: University of Arizona

**Step 1** Enter the travel times in L1 and the distances in L2.

**KEYSTROKES:** Refer to page 92 to review how to enter lists.

**Step 2** Graph the scatter plot.

**KEYSTROKES:** Refer to page 92 to review how to graph a scatter plot.

**Step 3** Compute and graph the equation for the curve of best fit. A quartic curve is the best fit for these data. You can verify this by comparing the $R^2$ values for each type of graph.

**KEYSTROKES:**  STAT 7 2nd [L1] , 2nd [L2] ENTER Y= VARS 5 1 GRAPH

**Step 4** Use the [CALC] feature to find the value of the function for $x = 8.5$.

**KEYSTROKES:** 2nd [CALC] 1 8.5 ENTER

After 8.5 minutes, you would expect the wave to be felt approximately 5000 kilometers away.

**Exercises**

For Exercises 1–3, use the table that shows how many minutes out of each eight-hour workday are used to pay one day’s worth of taxes.

1. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of minutes to the number of years since 1930. Try LinReg, QuadReg, and CubicReg.

2. Write the equation for the curve that best fits the data.

3. Based on this equation, how many minutes should you expect to work each day in the year 2010 to pay one day’s taxes?

<table>
<thead>
<tr>
<th>Year</th>
<th>Minutes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1940</td>
<td>83</td>
</tr>
<tr>
<td>1950</td>
<td>117</td>
</tr>
<tr>
<td>1960</td>
<td>130</td>
</tr>
<tr>
<td>1970</td>
<td>141</td>
</tr>
<tr>
<td>1980</td>
<td>145</td>
</tr>
<tr>
<td>1990</td>
<td>145</td>
</tr>
<tr>
<td>2000</td>
<td>160</td>
</tr>
</tbody>
</table>

Source: Tax Foundation

Other Calculator Keystrokes at algebra2.com
For Exercises 4–7, use the table that shows the estimated number of alternative-fueled vehicles in use in the United States per year.

4. Draw a scatter plot of the data. Then graph several curves of best fit that relate the number of vehicles to the year. Try LinReg, QuadReg, and CubicReg. (Hint: Enter the x-values as years since 1994.)

5. Write the equation for the curve that best fits the data. Round to the nearest tenth.

6. Based on this equation before rounding, how many Alternative-Fueled Vehicles would you expect to be in use in the year 2008?

7. Find a curve of best fit that is quartic. Is it a better fit than the equation you wrote in Exercise 5? Explain.

For Exercises 8–11, use the table that shows the distance from the Sun to the Earth for each month of the year.

8. Draw a scatter plot of the data. Then graph several curves of best fit that relate the distance to the month. Try LinReg, QuadReg, and CubicReg.

9. Write the equation for the curve that best fits the data.

10. Based on this equation, what is the distance from the Sun to the Earth halfway through September?

11. Would you use this model to find the distance from the Sun to Earth in subsequent years? Explain your reasoning.

EXTENSION

For Exercises 12–15, design and complete your own data analysis.

12. Write a question that could be answered by examining data. For example, you might estimate the number of people living in your town 5 years from now or predict the future cost of a car.

13. Collect and organize the data you need to answer the question you wrote. You may need to research your topic on the internet or conduct a survey to collect the data you need.

14. Make a scatter plot and find a regression equation for your data. Then use the regression equation to answer the question.
Simplify. Assume that no variable equals 0. (Lesson 6-1)

1. \((-3x^2y)^3 (2x)^2\)  
2. \(\frac{a^5b^{-2}c}{a^3b^2c^4}\)  
3. \(\left(\frac{x^2z}{xz^4}\right)^2\)

4. **CHEMISTRY** One gram of water contains about \(3.34 \times 10^{22}\) molecules. About how many molecules are in \(5 \times 10^2\) grams of water? (Lesson 6-1)

Simplify. (Lesson 6-2)

5. \((9x + 2y) - (7x - 3y)\)  
6. \((t + 2)(3t - 4)\)  
7. \((n + 2)(n^2 - 3n + 1)\)  
8. \(4a(ab + 5a^2)\)

9. **MULTIPLE CHOICE** The area of the base of a rectangular suitcase measures \(3x^2 + 5x - 4\) square units. The height of the suitcase measures \(2x\) units. Which polynomial expression represents the volume of the suitcase? (Lesson 6-2)

A. \(3x^3 + 5x^2 - 4x\)  
B. \(6x^2 + 10x - 8\)  
C. \(6x^3 + 10x^2 - 8x\)  
D. \(3x^3 + 10x^2 - 4\)

Simplify. (Lesson 6-3)

10. \((m^3 - 4m^2 - 3m - 7) ÷ (m - 4)\)  
11. \(\frac{2d^3 - d^2 - 9d + 9}{2d - 3}\)  
12. \((x^3 + x^2 - 13x - 28) ÷ (x - 4)\)  
13. \(\frac{3y^3 + 7y^2 - y - 5}{y + 2}\)

14. **WOODWORKING** Arthur is building a rectangular table with an area of \(3x^2 - 17x - 28\) square feet. If the length of the table is \(3x + 4\) feet, what should the width of the rectangular table be? (Lesson 6-3)

15. **PETS** A pet food company estimates that it costs \(0.02x^2 + 3x + 250\) dollars to produce \(x\) bags of dog food. Find an expression for the average cost per unit. (Lesson 6-3)

16. If \(p(x) = 2x^3 - x\), find \(p(a - 1)\). (Lesson 6-4)

17. Describe the end behavior of the graph. Then determine whether it represents an odd-degree or an even-degree polynomial function and state the number of real zeroes. (Lesson 6-4)

18. **WIND CHILL** The function \(C(s) = 0.013s^2 - s - 7\) estimates the wind chill temperature \(C(s)\) at \(0^\circ F\) for wind speeds \(s\) from 5 to 30 miles per hour. Estimate the wind chill temperature at \(0^\circ F\) if the wind speed is 27 miles per hour. (Lesson 6-4)

19. The formula \(L(t) = \frac{8t^2}{\pi^2}\) represents the swing of a pendulum. \(L\) is the length of the pendulum in feet, and \(t\) is the time in seconds to swing back and forth. Find the length of a pendulum \(L(t)\) that makes one swing in 2 seconds. (Lesson 6-4)

20. **MULTIPLE CHOICE** The function \(f(x) = x^2 - 4x + 3\) has a relative minimum located at which of the following \(x\)-values? (Lesson 6-5)

F. \(-2\)  
H. \(3\)  
G. \(2\)  
J. \(4\)

21. Graph \(y = x^3 + 2x^2 - 4x - 6\). Estimate the \(x\)-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

22. **MARKET PRICE** Prices of oranges from January to August can be modeled by \((1, 2.7), (2, 4.4), (3, 4.9), (4, 5.5), (5, 4.3), (6, 5.3), (7, 3.5), (8, 3.9)\). How many turning points would the graph of a polynomial function through these points have? Describe them. (Lesson 6-5)
The Taylor Manufacturing Company makes open metal boxes of various sizes. Each sheet of metal is 50 inches long and 32 inches wide. To make a box, a square is cut from each corner.

The volume of the box depends on the side length \( x \) of the cut squares. It is given by \( V(x) = 4x^3 - 164x^2 + 1600x \). You can solve a polynomial equation to find the dimensions of the square to cut for a box with specific volume.

**Factor Polynomials** Whole numbers are factored using prime numbers. For example, \( 100 = 2 \cdot 2 \cdot 5 \cdot 5 \). Many polynomials can also be factored. Their factors, however, are other polynomials. Polynomials that cannot be factored are called *prime*. One method for finding the dimensions of the square to cut to make a box involves factoring the polynomial that represents the volume.

The table below summarizes the most common factoring techniques used with polynomials. Some of these techniques were introduced in Lesson 5-3. The others will be presented in this lesson.

<table>
<thead>
<tr>
<th>Number of Terms</th>
<th>Factoring Technique</th>
<th>General Case</th>
</tr>
</thead>
<tbody>
<tr>
<td>any number</td>
<td>Greatest Common Factor (GCF)</td>
<td>( a^3b^2 + 2a^2b - 4ab^2 = ab(a^2b + 2a - 4b) )</td>
</tr>
<tr>
<td>two</td>
<td>Difference of Two Squares</td>
<td>( a^2 - b^2 = (a + b)(a - b) )</td>
</tr>
<tr>
<td></td>
<td>Sum of Two Cubes</td>
<td>( a^3 + b^3 = (a + b)(a^2 - ab + b^2) )</td>
</tr>
<tr>
<td></td>
<td>Difference of Two Cubes</td>
<td>( a^3 - b^3 = (a - b)(a^2 + ab + b^2) )</td>
</tr>
<tr>
<td>three</td>
<td>Perfect Square Trinomials</td>
<td>( a^2 + 2ab + b^2 = (a + b)^2 )</td>
</tr>
<tr>
<td></td>
<td></td>
<td>( a^2 - 2ab + b^2 = (a - b)^2 )</td>
</tr>
<tr>
<td></td>
<td>General Trinomials</td>
<td>( ax^2 + (ad + bc)x + bd = (ax + b)(cx + d) )</td>
</tr>
<tr>
<td>four or more</td>
<td>Grouping</td>
<td>( ax + bx + ay + by = x(a + b) + y(a + b) = (a + b)(x + y) )</td>
</tr>
</tbody>
</table>

Whenever you factor a polynomial, always look for a common factor first. Then determine whether the resulting polynomial factor can be factored again using one or more of the methods listed above.
Factor $6x^2y^2 - 2xy^2 + 6x^3y$.

$6x^2y^2 - 2xy^2 + 6x^3y = (2 \cdot 3 \cdot x \cdot x \cdot y \cdot y) - (2 \cdot x \cdot y \cdot y) + (2 \cdot 3 \cdot x \cdot x \cdot x \cdot y)$

$= (2xy \cdot 3xy) - (2xy \cdot y) + (2xy \cdot 3x^2)$

$= 2xy(3xy - y + 3x^2)$

The GCF is $2xy$. The remaining polynomial cannot be factored using the methods above.

CHECK Your Progress
Factor completely.

1A. $18x^3y^4 - 12x^2y^3 - 6xy^2$

1B. $a^4b^4 + 3a^3b^4 + a^2b^3$

Factor $a^3 - 4a^2 + 3a - 12$.

$a^3 - 4a^2 + 3a - 12 = (a^3 - 4a^2) + (3a - 12)$

$= a^2(a - 4) + 3(a - 4)$

$= (a - 4)(a^2 + 3)$

Factor completely.

2A. $x^2 + 3xy + 2xy^2 + 6y^3$

2B. $6a^3 - 9a^2b + 4ab - 6b^2$

Factoring by grouping is the only method that can be used to factor polynomials with four or more terms. For polynomials with two or three terms, it may be possible to factor the polynomial according to one of the patterns shown on page 349.

Factor each polynomial.

3a. $8x^3 - 24x^2 + 18x$

This trinomial does not fit any of the factoring patterns. First, factor out the GCF. Then the remaining trinomial is a perfect square trinomial.

$8x^3 - 24x^2 + 18x = 2x(4x^2 - 12x + 9)$

$= 2x(2x - 3)^2$

Perfect square trinomial

3b. $m^6 - n^6$

This polynomial could be considered the difference of two squares or the difference of two cubes. The difference of two squares should always be done before the difference of two cubes. This will make the next step of the factorization easier.

$m^6 - n^6 = (m^3 + n^3)(m^3 - n^3)$

$= (m + n)(m^2 - mn + n^2)(m - n)(m^2 + mn + n^2)$

Difference of two squares

Sum and difference of two cubes

CHECK Your Progress

3A. $3xy^2 - 48x$

3B. $c^3d^3 + 27$
You can use a graphing calculator to check that the factored form of a polynomial is correct.

**GRAPHING CALCULATOR LAB**

**Factoring Polynomials**

Is the factored form of $2x^2 - 11x - 21$ equal to $(2x - 7)(x + 3)$? You can find out by graphing $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$. If the two graphs coincide, the factored form is probably correct.

- Enter $y = 2x^2 - 11x - 21$ and $y = (2x - 7)(x + 3)$ on the Y= screen.
- Graph the functions. Since two different graphs appear, $2x^2 - 11x - 21 \neq (2x - 7)(x + 3)$.

**THINK AND DISCUSS**

1. Determine if $x^2 + 5x - 6 = (x - 3)(x - 2)$ is a true statement. If not, write the correct factorization.

2. Does this method guarantee a way to check the factored form of a polynomial? Why or why not?

In some cases, you can rewrite a polynomial in $x$ in the form $au^2 + bu + c$. For example, by letting $u = x^2$, the expression $x^4 - 16x^2 + 60$ can be written as $(x^2)^2 - 16(x^2) + 60$ or $u^2 - 16u + 60$. This new, but equivalent, expression is said to be in **quadratic form**.

**KEY CONCEPT**

**Quadratic Form**

An expression that is quadratic in form can be written as $au^2 + bu + c$ for any numbers $a, b,$ and $c$, $a \neq 0$, where $u$ is some expression in $x$. The expression $au^2 + bu + c$ is called the quadratic form of the original expression.

**EXAMPLE**

**Write Expressions in Quadratic Form**

Write each expression in quadratic form, if possible.

4. $x^4 + 13x^2 + 36$
   
   $x^4 + 13x^2 + 36 = (x^2)^2 + 13(x^2) + 36 \quad (x^2)^2 = x^4$

b. $12x^8 - x^2 + 10$
   
   This cannot be written in quadratic form since $x^8 \neq (x^2)^2$.

**CHECK Your Progress**

4A. $16x^6 - 625$
4B. $9x^{10} - 15x^4 + 9$

**Solve Equations Using Quadratic Form**

In Chapter 5, you learned to solve quadratic equations by factoring and using the Zero Product Property. You can extend these techniques to solve higher-degree polynomial equations.

**Extra Examples at** [algebra2.com](http://algebra2.com)
EXAMPLE

Solve Polynomial Equations

Solve each equation.

a. \( x^4 - 13x^2 + 36 = 0 \)

\[
\begin{align*}
&\quad \text{Original equation} \\
(x^2 - 9)(x^2 - 4) = 0 &\text{Write the expression on the left in quadratic form.} \\
(x - 3)(x + 3)(x - 2)(x + 2) = 0 &\text{Factor the trinomial.} \\
\text{Factor each difference of squares.} \\
\end{align*}
\]

Use the Zero Product Property.

\[
\begin{align*}
x - 3 = 0 & \quad \text{or} \quad x + 3 = 0 \quad \text{or} \quad x - 2 = 0 \quad \text{or} \quad x + 2 = 0 \\
x = 3 & \quad x = -3 & \quad x = 2 & \quad x = -2
\end{align*}
\]

The solutions are \(-3, -2, 2, \) and \(3.\)

CHECK The graph of \( f(x) = x^4 - 13x^2 + 36 \)

intersects the \(x\)-axis at \(-3, -2, 2, \) and \(3.\)

b. \( x^3 + 343 = 0 \)

\[
\begin{align*}
x^3 + 343 &= 0 &\text{Original equation} \\
(x)^3 + 7^3 &= 0 &\text{This is the sum of two cubes.} \\
(x + 7)(x^2 - x(7) + 7^2) &= 0 &\text{Sum of two cubes formula with} \ a = x \text{ and } b = 7 \\
(x + 7)(x^2 - 7x + 49) &= 0 &\text{Simplify.} \\
(x + 7) &= 0 \text{ or } x^2 - 7x + 49 &= 0 &\text{Zero Product Property} \\
(x + 7) &= 0 \text{ or } x^2 - 7x + 49 &= 0 \\
x &= -7 &\text{ or } x^2 - 7x + 49 &= 0
\end{align*}
\]

The solution of the first equation is \(-7.\) The second equation can be solved by using the Quadratic Formula.

\[
\begin{align*}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &\text{Quadratic Formula} \\
&= \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(49)}}{2(1)} &\text{Replace } a \text{ with } 1, \text{ } b \text{ with } -7, \text{ and } c \text{ with } 49. \\
&= \frac{7 \pm \sqrt{-147}}{2} &\text{Simplify.} \\
&= \frac{7 \pm 7i\sqrt{3}}{2} &\sqrt{147} \times \sqrt{-1} = 7i\sqrt{3}
\end{align*}
\]

Thus, the solutions of the original equation are \(-7, \frac{7 + 7i\sqrt{3}}{2}, \) and \(\frac{7 - 7i\sqrt{3}}{2}.\)

CHECK The graph of \( f(x) = x^3 + 343 \)

confirms the solution.

5A. \( x^4 - 29x^2 + 100 = 0 \) 5B. \( x^3 + 8 = 0 \)

Personal Tutor at algebra2.com
Factor completely. If the polynomial is not factorable, write prime.

1. \(-12x^2 - 6x\) 
2. \(a^2 + 5a + ab\) 
3. \(21 - 7y + 3x - xy\) 
4. \(y^2 + 4y + 2y + 8\) 
5. \(z^2 - 4z - 12\) 
6. \(3b^2 - 48\) 
7. \(16w^2 - 169\) 
8. \(h^3 + 8000\)

Write each expression in quadratic form, if possible.

9. \(5y^4 + 7y^3 - 8\) 
10. \(84n^4 - 62n^2\)

Solve each equation.

11. \(x^4 - 50x^2 + 49 = 0\) 
12. \(x^3 - 125 = 0\) 
13. POOL The Shelby University swimming pool is in the shape of a rectangular prism and has a volume of 28,000 cubic feet. The dimensions of the pool are \(x\) feet deep by \(7x - 6\) feet wide by \(9x - 2\) feet long. How deep is the pool?

Factor completely. If the polynomial is not factorable, write prime.

14. \(2xy^3 - 10x\) 
15. \(6a^2b^2 + 18ab^3\) 
16. \(12cd^3 - 8c^2d^2 + 10c^3d^3\) 
17. \(3a^2bx + 15cx^2y + 25ad^3y\) 
18. \(8yz - 6z - 12y + 9\) 
19. \(3ax - 15a + x - 5\) 
20. \(y^2 - 5y + 4\) 
21. \(2b^2 + 13b - 7\) 
22. \(z^3 + 125\) 
23. \(t^3 - 8\)

Write each expression in quadratic form, if possible.

24. \(2x^4 + 6x^2 - 10\) 
25. \(a^8 + 10a^2 - 16\) 
26. \(11y^6 + 44n^3\) 
27. \(7b^5 - 4b^3 + 2b\) 
28. \(7x^9 - 3x^{3/2} + 4\) 
29. \(6x^{5/2} - 4x^{3/2} - 16\)

Solve each equation.

30. \(x^4 - 34x^2 + 225 = 0\) 
31. \(x^4 - 15x^2 - 16 = 0\) 
32. \(x^4 + 6x^2 - 27 = 0\) 
33. \(x^3 + 64 = 0\) 
34. \(27x^3 + 1 = 0\) 
35. \(8x^3 - 27 = 0\)

DESIGN For Exercises 36–38, use the following information.

Jill is designing a picture frame for an art project. She plans to have a square piece of glass in the center and surround it with a decorated ceramic frame, which will also be a square. The dimensions of the glass and frame are shown in the diagram at the right. Jill determines that she needs 27 square inches of material for the frame.

36. Write a polynomial equation that models the area of the frame.
37. What are the dimensions of the glass piece?
38. What are the dimensions of the frame?
39. **GEOMETRY** The width of a rectangular prism is \(w\) centimeters. The height is 2 centimeters less than the width. The length is 4 centimeters more than the width. If the volume of the prism is 8 times the measure of the length, find the dimensions of the prism.

40. Find the factorization of \(3x^2 + x - 2\).

41. What are the factors of \(2y^2 + 9y + 4\)?

**Factor completely. If the polynomial is not factorable, write prime.**

42. \(3n^2 + 21n - 24\)

43. \(y^4 - z^2\)

44. \(16a^2 + 25b^2\)

45. \(3x^2 - 27y^2\)

46. \(x^4 - 81\)

47. \(3a^3 + 2a^2 - 5a + 9a^2b + 6ab - 15b\)

**PACKAGING** For Exercises 48 and 49, use the following information.

A computer manufacturer needs to change the dimensions of its foam packaging for a new model of computer. The width of the original piece is three times the height, and the length is equal to the height squared. The volume of the new piece can be represented by the equation \(V(h) = 3h^4 + 11h^3 + 18h^2 + 44h + 24\), where \(h\) is the height of the original piece.

48. Factor the equation for the volume of the new piece to determine three expressions that represent the height, length, and width of the new piece.

49. How much did each dimension of the packaging increase for the new foam piece?

50. **LANDSCAPING** A boardwalk that is \(x\) feet wide is built around a rectangular pond. The pond is 30 feet wide and 40 feet long. The combined area of the pond and the boardwalk is 2000 square feet. What is the width of the boardwalk?

**CHECK FACTORING** Use a graphing calculator to determine if each polynomial is factored correctly. Write yes or no. If the polynomial is not factored correctly, find the correct factorization.

51. \(3x^2 + 5x + 2 \neq (3x + 2)(x + 1)\)

52. \(x^3 + 8 \neq (x + 2)(x^2 - x + 4)\)

53. \(2x^2 - 5x - 3 \neq (x - 1)(2x + 3)\)

54. \(3x^2 - 48 \neq 3(x + 4)(x - 4)\)

55. **OPEN ENDED** Give an example of an equation that is not quadratic but can be written in quadratic form. Then write it in quadratic form.

56. **CHALLENGE** Factor \(64p^{2n} + 16p^n + 1\).

57. **REASONING** Find a counterexample to the statement \(a^2 + b^2 = (a + b)^2\).

58. **CHALLENGE** Explain how you would solve \(|a - 3| - 9|a - 3| = -8\). Then solve the equation.

59. **Writing in Math** Use the information on page 349 to explain how solving a polynomial equation can help you find dimensions. Explain how you could determine the dimensions of the cut square if the desired volume was 3600 cubic inches. Explain why there can be more than one square that can be cut to produce the same volume.
60. **ACT/SAT** Which is not a factor of \( x^3 - x^2 - 2x \)?
   A. \( x \)  
   B. \( x + 1 \)  
   C. \( x - 1 \)  
   D. \( x - 2 \)

61. **ACT/SAT** The measure of the largest angle of a triangle is 14 less than twice the measure of the smallest angle. The third angle is 2 more than the measure of the smallest angle. What is the measure of the smallest angle?
   F. 46  G. 48  H. 50  J. 82

---

**Spiral Review**

Graph each polynomial function. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur. (Lesson 6-5)

63. \( f(x) = x^3 - 6x^2 + 4x + 3 \)  
64. \( f(x) = -x^4 + 2x^3 + 3x^2 - 7x + 4 \)

Find \( p(7) \) and \( p(-3) \) for each function. (Lesson 6-4)

65. \( p(x) = x^2 - 5x + 3 \)  
66. \( p(x) = x^3 - 11x - 4 \)  
67. \( p(x) = \frac{2}{3}x^4 - 3x^3 \)

68. **PHOTOGRAPHY** The perimeter of a rectangular picture is 86 inches. Twice the width exceeds the length by 2 inches. What are the dimensions of the picture? (Lesson 3-2)

Determine whether each relation is a function. Write yes or no. (Lesson 2-1)

69. 

70. 

---

**GET READY for the Next Lesson**

**PREREQUISITE SKILL** Find each quotient. (Lesson 6-3)

71. \( (x^3 + 4x^2 - 9x + 4) \div (x - 1) \)  
72. \( (4x^3 - 8x^2 - 5x - 10) \div (x + 2) \)  
73. \( (x^4 - 9x^2 - 2x + 6) \div (x - 3) \)  
74. \( (x^4 + 3x^3 - 8x^2 + 5x - 6) \div (x + 1) \)
The number of international travelers to the United States since 1986 can be modeled by the equation
\[ T(x) = 0.02x^3 - 0.6x^2 + 6x + 25.9, \]
where \( x \) is the number of years since 1986 and \( T(x) \) is the number of travelers in millions. To estimate the number of travelers in 2006, you can evaluate the function by substituting 20 for \( x \), or you can use synthetic substitution.

### Synthetic Substitution
Synthetic division can be used to find the value of a function. Consider the polynomial function
\[ f(a) = 4a^2 - 3a + 6. \]
Divide the polynomial by \( a - 2 \).

#### Method 1  Long Division
\[
\begin{array}{c|cc}
  & 4a & + 5 \\
\hline
a - 2 & 4a^2 & - 3a & + 6 \\
   & 4a^2 & - 8a & \\
   & 5a & + 6 & \\
   & 5a & - 10 & \\
   & 16 & & \\
\end{array}
\]

Compare the remainder of 16 to \( f(2) \).

\[
f(2) = 4(2)^2 - 3(2) + 6 \]
\[
= 16 - 6 + 6 \quad \text{Replace } a \text{ with } 2.
\]
\[
= 16 \quad \text{Multiply.}
\]
\[
= 16 \quad \text{Simplify.}
\]

Notice that the value of \( f(2) \) is the same as the remainder when the polynomial is divided by \( a - 2 \). This illustrates the **Remainder Theorem**.

### Remainder Theorem

\[
\begin{array}{c|cc}
\text{Dividend} & f(x) \quad \text{equals} \quad q(x) \quad \text{times} \quad (x - a) \quad \text{plus} \quad f(a),
\end{array}
\]

Where \( q(x) \) is a polynomial with degree one less than the degree of \( f(x) \).

When synthetic division is used to evaluate a function, it is called **synthetic substitution**. It is a convenient way of finding the value of a function, especially when the degree of the polynomial is greater than 2.
EXAMPLE

Synthetic Substitution

If \( f(x) = 2x^4 - 5x^2 + 8x - 7 \), find \( f(6) \).

Method 1 Synthetic Substitution

By the Remainder Theorem, \( f(6) \) should be the remainder when you divide the polynomial by \( x - 6 \).

\[
\begin{array}{c|cccc}
6 & 2 & 0 & -5 & 8 & -7 \\
 & & 12 & 72 & 402 & 2460 \\
\hline
 & 2 & 12 & 67 & 410 & 2453 \\
\end{array}
\]

Notice that there is no \( x^3 \) term. A zero is placed in this position as a placeholder.

The remainder is 2453. Thus, by using synthetic substitution, \( f(6) = 2453 \).

Method 2 Direct Substitution

Replace \( x \) with 6.

\[
f(x) = 2x^4 - 5x^2 + 8x - 7 \quad \text{Original function}
\]

\[
f(6) = 2(6)^4 - 5(6)^2 + 8(6) - 7 \quad \text{Replace } x \text{ with } 6.
\]

\[
= 2592 - 180 + 48 - 7 \quad \text{or} \quad 2453 \quad \text{Simplify.}
\]

By using direct substitution, \( f(6) = 2453 \). Both methods give the same result.

CHECK Your Progress

1A. If \( f(x) = 3x^3 - 6x^2 + x - 11 \), find \( f(3) \).

1B. If \( g(x) = 4x^5 + 2x^3 + x^2 - 1 \), find \( f(-1) \).

Factors of Polynomials

The synthetic division below shows that the quotient of \( x^4 + x^3 - 17x^2 - 20x + 32 \) and \( x - 4 \) is \( x^3 + 5x^2 + 3x - 8 \).

\[
\begin{array}{c|cccc}
4 & 1 & 1 & -17 & -20 & 32 \\
 & & 4 & 20 & 12 & -32 \\
\hline
 & 1 & 5 & 3 & -8 & 0 \\
\end{array}
\]

When you divide a polynomial by one of its binomial factors, the quotient is called a depressed polynomial. From the results of the division and by using the Remainder Theorem, we can make the following statement.

\[
x^4 + x^3 - 17x^2 - 20x + 32 = (x^3 + 5x^2 + 3x - 8) \cdot (x - 4) + 0
\]

Since the remainder is 0, \( f(4) = 0 \). This means that \( x - 4 \) is a factor of \( x^4 + x^3 - 17x^2 - 20x + 32 \). This illustrates the Factor Theorem, which is a special case of the Remainder Theorem.

KEY CONCEPT

The binomial \( x - a \) is a factor of the polynomial \( f(x) \) if and only if \( f(a) = 0 \).

If \( x - a \) is a factor of \( f(x) \), then \( f(a) \) has a factor of \( (a - a) \), or 0. Since a factor of \( f(a) \) is 0, \( f(a) = 0 \). Now assume that \( f(a) = 0 \). If \( f(a) = 0 \), then the Remainder Theorem states that the remainder is 0 when \( f(x) \) is divided by \( x - a \). This means that \( x - a \) is a factor of \( f(x) \). This proves the Factor Theorem.
Suppose you wanted to find the factors of \( x^3 - 3x^2 - 6x + 8 \). One approach is to graph the related function, \( f(x) = x^3 - 3x^2 - 6x + 8 \). From the graph, you can see that the graph of \( f(x) \) crosses the \( x \)-axis at \(-2\), \(1\), and \(4\). These are the zeros of the function. Using these zeros and the Zero Product Property, we can express the polynomial in factored form.

\[
f(x) = [x - (-2)](x - 1)(x - 4) \\
     = (x + 2)(x - 1)(x - 4)
\]

This method of factoring a polynomial has its limitations. Most polynomial functions are not easily graphed, and once graphed, the exact zeros are often difficult to determine.

**EXAMPLE**

**Use the Factor Theorem**

Show that \( x + 3 \) is a factor of \( x^3 + 6x^2 - x - 30 \). Then find the remaining factors of the polynomial.

The binomial \( x + 3 \) is a factor of the polynomial if \(-3\) is a zero of the related polynomial function. Use the Factor Theorem and synthetic division.

\[
\begin{array}{c|cccc}
-3 & 1 & 6 & -1 & -30 \\
 & & -3 & -9 & 30 \\
\hline
 & 1 & 3 & -10 & 0
\end{array}
\]

Since the remainder is 0, \( x + 3 \) is a factor of the polynomial. The polynomial \( x^3 + 6x^2 - x - 30 \) can be factored as \((x + 3)(x^2 + 3x - 10)\). The polynomial \( x^2 + 3x - 10 \) is the depressed polynomial. Check to see if this polynomial can be factored.

\[
x^2 + 3x - 10 = (x - 2)(x + 5) \quad \text{Factor the trinomial.}
\]

So, \( x^3 + 6x^2 - x - 30 = (x + 3)(x - 2)(x + 5) \).

**Check Your Progress**

2. Show that \( x - 2 \) is a factor of \( x^3 - 7x^2 + 4x + 12 \). Then find the remaining factors of the polynomial.

**EXAMPLE**

**Find All Factors**

**GEOMETRY** The volume of the rectangular prism is given by \( V(x) = x^3 + 3x^2 - 36x + 32 \). Find the missing measures.

The volume of a rectangular prism is \( \ell \times w \times h \).

You know that one measure is \( x - 4 \), so \( x - 4 \) is a factor of \( V(x) \).

\[
\begin{array}{c|cccc}
4 & 1 & 3 & -36 & 32 \\
 & & 4 & 28 & -32 \\
\hline
 & 1 & 7 & -8 & 0
\end{array}
\]

The quotient is \( x^2 + 7x - 8 \). Use this to factor \( V(x) \).
Exercises

30–33
18–29
10–17

For Examples 2, 3 (pp. 358–359)
Example 1

Use synthetic substitution to find \( f(3) \) and \( f(-4) \) for each function.

1. \( f(x) = x^3 - 2x^2 - x + 1 \)
2. \( f(x) = 5x^4 - 6x^2 + 2 \)

For Exercises 3–5, use the following information.
The projected sales of e-books in millions of dollars can be modeled by the function \( S(x) = -17x^3 + 200x^2 - 113x + 44 \), where \( x \) is the number of years since 2000.
3. Use synthetic substitution to estimate the sales for 2008.
4. Use direct substitution to evaluate \( S(8) \).
5. Which method—synthetic substitution or direct substitution—do you prefer to use to evaluate polynomials? Explain your answer.

Examples 2, 3 (pp. 358–359)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

6. \( x^3 - x^2 - 5x - 3; x + 1 \)
7. \( x^3 - 3x + 2; x - 1 \)
8. \( 6x^3 - 25x^2 + 2x + 8; 3x - 2 \)
9. \( x^4 + 2x^3 - 8x - 16; x + 2 \)

Use synthetic substitution to find \( g(3) \) and \( g(-4) \) for each function.

10. \( g(x) = x^2 - 8x + 6 \)
11. \( g(x) = x^3 + 2x^2 - 3x + 1 \)
12. \( g(x) = x^3 - 5x + 2 \)
13. \( g(x) = x^4 - 6x - 8 \)
14. \( g(x) = 2x^3 - 8x^2 - 2x + 5 \)
15. \( g(x) = 3x^4 + x^3 - 2x^2 + x + 12 \)
16. \( g(x) = x^5 + 8x^3 + 2x - 15 \)
17. \( g(x) = x^6 - 4x^4 + 3x^2 - 10 \)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

18. \( x^3 + 2x^2 - x - 2; x - 1 \)
19. \( x^3 - x^2 - 10x - 8; x + 1 \)
20. \( x^3 + x^2 - 16x - 16; x + 4 \)
21. \( x^3 - 6x^2 + 11x - 6; x - 2 \)
22. \( 2x^3 - 5x^2 - 28x + 15; x - 5 \)
23. \( 3x^3 + 10x^2 - x - 12; x + 3 \)
24. \( 2x^3 + 7x^2 - 53x - 28; 2x + 1 \)
25. \( 2x^3 + 17x^2 + 23x - 42; 2x + 7 \)
26. \( x^4 + 2x^3 + 2x^2 - 2x - 3; x + 1 \)
27. \( 16x^5 - 32x^4 - 81x + 162; x - 2 \)
28. Use synthetic substitution to show that \( x - 8 \) is a factor of \( x^3 - 4x^2 - 29x - 24 \). Then find any remaining factors.
29. Use the graph of the polynomial function at the right to determine at least one binomial factor of the polynomial. Then find all the factors of the polynomial.

BOATING For Exercises 30 and 31, use the following information.
A motor boat traveling against waves accelerates from a resting position. Suppose the speed of the boat in feet per second is given by the function \( f(t) = -0.04t^4 + 0.8t^3 + 0.5t^2 - t \), where \( t \) is the time in seconds.

30. Find the speed of the boat at 1, 2, and 3 seconds.
31. It takes 6 seconds for the boat to travel between two buoys while it is accelerating. Use synthetic substitution to find \( f(6) \) and explain what this means.

ENGINEERING For Exercises 32 and 33, use the following information.
When a certain type of plastic is cut into sections, the length of each section determines its strength. The function \( f(x) = x^4 - 14x^3 + 69x^2 - 140x + 100 \) can describe the relative strength of a section of length \( x \) feet. Sections of plastic \( x \) feet long, where \( f(x) = 0 \), are extremely weak. After testing the plastic, engineers discovered that sections 5 feet long were extremely weak.

32. Show that \( x - 5 \) is a factor of the polynomial function.
33. Are there other lengths of plastic that are extremely weak? Explain your reasoning.

Find values of \( k \) so that each remainder is 3.

34. \((x^2 - x + k) \div (x - 1)\)
35. \((x^2 + kx - 17) \div (x - 2)\)
36. \((x^2 + 5x + 7) \div (x + k)\)
37. \((x^3 + 4x^2 + x + k) \div (x + 2)\)

PERSONAL FINANCE For Exercises 38–41, use the following information.
Zach has purchased some home theater equipment for $2000, which he is financing through the store. He plans to pay $340 per month and wants to have the balance paid off after six months. The formula \( B(x) = 2000x^6 - 340(x^3 + x^4 + x^3 + x^2 + x + 1) \) represents his balance after six months if \( x \) represents 1 plus the monthly interest rate (expressed as a decimal).

38. Find his balance after 6 months if the annual interest rate is 12%. (Hint: The monthly interest rate is the annual rate divided by 12, so \( x = 1.01 \).)
39. Find his balance after 6 months if the annual interest rate is 9.6%.
40. How would the formula change if Zach wanted to pay the balance in five months?
41. Suppose he finances his purchase at 10.8% and plans to pay $410 every month. Will his balance be paid in full after five months?

42. OPEN ENDED Give an example of a polynomial function that has a remainder of 5 when divided by \( x - 4 \).
43. REASONING Determine the dividend, divisor, quotient, and remainder represented by the synthetic division at the right.
**Lesson 6-7 The Remainder and Factor Theorems**

**44. CHALLENGE** Consider the polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where $a + b + c + d + e = 0$. Show that this polynomial is divisible by $x - 1$.

**45. Writing in Math** Use the information on page 356 to explain how you can use the Remainder Theorem to evaluate polynomials. Include an explanation of when it is easier to use the Remainder Theorem to evaluate a polynomial rather than substitution. Evaluate the expression for the number of international travelers to the U.S. for $x = 20$.

---

**STANDARDIZED TEST PRACTICE**

**46. ACT/SAT** Use the graph of the polynomial function at the right. Which is not a factor of the polynomial $x^5 + x^4 - 3x^3 - 3x^2 - 4x - 4$?

- A $(x - 2)$
- B $(x + 2)$
- C $(x - 1)$
- D $(x + 1)$

**47. REVIEW** The total area of a rectangle is $25a^4 - 16b^2$. Which factors could represent the length times width?

- F $(5a^2 + 4b)(5a^2 - 4b)$
- G $(5a^2 + 4b)(5a^2 + 4b)$
- H $(5a - 4b)(5a - 4b)$
- J $(5a + 4b)(5a - 4b)$

---

**Spiral Review**

Factor completely. If the polynomial is not factorable, write prime. (Lesson 6-6)

48. $7xy^3 - 14x^2y^5 + 28x^3y^2$

50. $2x^2 + 15x + 25$

49. $ab - 5a + 3b - 15$

51. $c^3 - 216$

Graph each function by making a table of values. (Lesson 6-5)

52. $f(x) = x^3 - 4x^2 + x + 5$

53. $f(x) = x^4 - 6x^3 + 10x^2 - x - 3$

54. **CITY PLANNING** City planners have laid out streets on a coordinate grid before beginning construction. One street lies on the line with equation $y = 2x + 1$. Another street that intersects the first street passes through the point $(2, -3)$ and is perpendicular to the first street. What is the equation of the line on which the second street lies? (Lesson 2-4)

---

**PREREQUISITE SKILL** Find the exact solutions of each equation by using the Quadratic Formula. (Lesson 5-6)

55. $x^2 + 7x + 8 = 0$

56. $3x^2 - 9x + 2 = 0$

57. $2x^2 + 3x + 2 = 0$
Main Ideas
- Determine the number and type of roots for a polynomial equation.
- Find the zeros of a polynomial function.

GET READY for the Lesson

When doctors prescribe medication, they give patients instructions as to how much to take and how often it should be taken. The amount of medication in your body varies with time. Suppose the equation \( M(t) = 0.5t^4 + 3.5t^3 - 100t^2 + 350t \) models the number of milligrams of a certain medication in the bloodstream \( t \) hours after it has been taken. The doctor can use the roots of this equation to determine how often the patient should take the medication to maintain a certain concentration in the body.

Types of Roots You have already learned that a zero of a function \( f(x) \) is any value \( c \) such that \( f(c) = 0 \). When the function is graphed, the real zeros of the function are the \( x \)-intercepts of the graph.

KEY CONCEPT

\[
\text{Let } f(x) = a_n x^n + \ldots + a_1 x + a_0 \text{ be a polynomial function. Then the following statements are equivalent.} \\
\begin{align*}
\bullet & \text{ } c \text{ is a zero of the polynomial function } f(x) . \\
\bullet & \text{ } x - c \text{ is a factor of the polynomial } f(x) . \\
\bullet & \text{ } c \text{ is a root or solution of the polynomial equation } f(x) = 0 .
\end{align*}
\]

In addition, if \( c \) is a real number, then \((c, 0)\) is an intercept of the graph of \( f(x) \).

The graph of \( f(x) = x^4 - 5x^2 + 4 \) is shown at the right. The zeros of the function are \(-2, -1, 1, \) and \(2\). The factors of the polynomial are \( x + 2, x + 1, x - 1, \) and \( x - 2 \). The solutions of the equation \( f(x) = 0 \) are \(-2, -1, 1, \) and \(2\). The \( x \)-intercepts of the graph of \( f(x) \) are \((-2, 0), (-1, 0), (1, 0), \) and \((2, 0)\).

When you solve a polynomial equation with degree greater than zero, it may have one or more real roots, or no real roots (the roots are imaginary numbers). Since real numbers and imaginary numbers both belong to the set of complex numbers, all polynomial equations with degree greater than zero will have at least one root in the set of complex numbers. This is the Fundamental Theorem of Algebra.

STUDY TIP

Look Back

For review of complex numbers, see Lesson 5-4.
EXAMPLE

Determine Number and Type of Roots

Solve each equation. State the number and type of roots.

a. \( x^2 - 8x + 16 = 0 \)
   \[
   x^2 - 8x + 16 = 0 \quad \text{Original equation}
   \]
   \[
   (x - 4)^2 = 0 \quad \text{Factor the left side as a perfect square trinomial.}
   \]
   \[
   x = 4 \quad \text{Solve for } x \text{ using the Square Root Property.}
   \]
   Since \( x - 4 \) is twice a factor of \( x^2 - 8x + 16 \), 4 is a double root. So this equation has one real repeated root, 4.

b. \( x^4 - 1 = 0 \)
   \[
   x^4 - 1 = 0
   \]
   \[
   (x^2 + 1)(x^2 - 1) = 0
   \]
   \[
   (x^2 + 1)(x + 1)(x - 1) = 0
   \]
   \[
   x^2 + 1 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad x - 1 = 0
   \]
   \[
   x^2 = -1 \quad x = -1 \quad x = 1
   \]
   \[
   x = \pm \sqrt{-1} \quad \text{or} \quad \pm i
   \]
   This equation has two real roots, 1 and \(-1\), and two imaginary roots, \(i\) and \(-i\).

CHECK Your Progress

1A. \( x^3 + 2x = 0 \)  
1B. \( x^4 - 16 = 0 \)

Compare the degree of each equation and the number of roots of each equation in Example 1. The following corollary of the Fundamental Theorem of Algebra is an even more powerful tool for problem solving.

**KEY CONCEPT**

A polynomial equation of the form \( P(x) = 0 \) of degree \( n \) with complex coefficients has exactly \( n \) roots in the set of complex numbers.

Similarly, a polynomial function of \( n \)th degree has exactly \( n \) zeros.

French mathematician René Descartes made more discoveries about zeros of polynomial functions. His rule of signs is given below.

**KEY CONCEPT**

Descartes’ Rule of Signs

If \( P(x) \) is a polynomial with real coefficients, the terms of which are arranged in descending powers of the variable,

- the number of positive real zeros of \( y = P(x) \) is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number, and
- the number of negative real zeros of \( y = P(x) \) is the same as the number of changes in sign of the coefficients of the terms of \( P(-x) \), or is less than this number by an even number.
Find Zeros

We can find all of the zeros of a function using some of the strategies you have already learned.

**EXAMPLE**

**Use Synthetic Substitution to Find Zeros**

Find all of the zeros of \( f(x) = x^3 - 4x^2 + 6x - 4 \).

Since \( f(x) \) has degree 3, the function has three zeros. To determine the possible number and type of real zeros, examine the number of sign changes for \( f(x) \) and \( f(-x) \).

\[
\begin{align*}
f(x) &= x^3 - 4x^2 + 6x - 4 \quad & f(-x) &= -x^3 - 4x^2 - 6x - 4
\end{align*}
\]

Since there are 3 sign changes for the coefficients of \( f(x) \), the function has 3 or 1 positive real zeros. Since there are no sign changes for the coefficient of \( f(-x) \), \( f(x) \) has no negative real zeros. Thus, \( f(x) \) has either 3 real zeros, or 1 real zero and 2 imaginary zeros.
To find these zeros, first list some possibilities and then eliminate those that are not zeros. Since none of the zeros are negative and \( f(0) = -4 \), begin by evaluating \( f(x) \) for positive integral values from 1 to 4. You can use a shortened form of synthetic substitution to find \( f(a) \) for several values of \( a \).

<table>
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<th>1</th>
<th>-4</th>
<th>6</th>
<th>-4</th>
</tr>
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<tr>
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<td>-3</td>
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<td>-1</td>
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<tr>
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<td>-2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
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<td>1</td>
<td>-1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0</td>
<td>6</td>
<td>20</td>
</tr>
</tbody>
</table>

Each row in the table shows the coefficients of the depressed polynomial and the remainder.

From the table, we can see that one zero occurs at \( x = 2 \). Since the depressed polynomial of this zero, \( x^2 - 2x + 2 \), is quadratic, use the Quadratic Formula to find the roots of the related quadratic equation, \( x^2 - 2x + 2 = 0 \).

\[
\begin{align*}
    x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
    &= \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} \\
    &= \frac{2 \pm \sqrt{-4}}{2} \\
    &= \frac{2 \pm 2i}{2} \\
    &= 1 \pm i
\end{align*}
\]

Thus, the function has one real zero at \( x = 2 \) and two imaginary zeros at \( x = 1 + i \) and \( x = 1 - i \). The graph of the function verifies that there is only one real zero.

3. Find all of the zeros of \( h(x) = x^3 + 2x^2 + 9x + 18 \).

In Chapter 5, you learned that solutions of a quadratic equation that contains imaginary numbers come in pairs. This applies to the zeros of polynomial functions as well. For any polynomial function with real coefficients, if an imaginary number is a zero of that function, its conjugate is also a zero. This is called the **Complex Conjugates Theorem**.

**KEY CONCEPT**

**Complex Conjugates Theorem**

Suppose \( a \) and \( b \) are real numbers with \( b \neq 0 \). If \( a + bi \) is a zero of a polynomial function with real coefficients, then \( a - bi \) is also a zero of the function.

**EXAMPLE**

Use Zeros to Write a Polynomial Function

4. Write a polynomial function of least degree with integral coefficients the zeros of which include 3 and \( 2 - i \).

**Explore** If \( 2 - i \) is a zero, then \( 2 + i \) is also a zero according to the Complex Conjugates Theorem. So, \( x - 3 \), \( x - (2 - i) \), and \( x - (2 + i) \) are factors of the polynomial function.
Plan  
Write the polynomial function as a product of its factors.
\[ f(x) = (x - 3)[x - (2 - i)][x - (2 + i)] \]

Solve  
Multiply the factors to find the polynomial function.
\[ f(x) = (x - 3)[x - (2 - i)][x - (2 + i)] \]
Write an equation. 
\[ = (x - 3)[(x - 2) + i][(x - 2) - i] \]
Regroup terms. 
\[ = (x - 3)((x - 2)^2 - i^2) \]
Rewrite as the difference of two squares. 
\[ = (x - 3)[x^2 - 4x + 4 - (-1)] \]
Square \( x - 2 \) and replace \( i^2 \) with \(-1\). 
\[ = (x - 3)(x^2 - 4x + 5) \]
Simplify. 
\[ = x^3 - 4x^2 + 5x - 3x^2 + 12x - 15 \]
Multiply using the Distributive Property. 
\[ = x^3 - 7x^2 + 17x - 15 \]
Combine like terms.

Check  
Since there are three zeros, the degree of the polynomial function must be three, so \( f(x) = x^3 - 7x^2 + 17x - 15 \) is a polynomial function of least degree with integral coefficients and zeros of 3, \( 2 - i \), and \( 2 + i \).

4. Write a polynomial function of least degree with integral coefficients the zeros of which include \(-1\) and \(1 + 2i\).
State the number of positive real zeros, negative real zeros, and imaginary zeros for each function.

17. \( f(x) = x^3 - 6x^2 + 1 \)
18. \( g(x) = 5x^3 + 8x^2 - 4x + 3 \)
19. \( h(x) = 4x^3 - 6x^2 + 8x - 5 \)
20. \( q(x) = x^4 + 5x^3 + 2x^2 - 7x - 9 \)
21. \( p(x) = x^5 - 6x^4 - 3x^3 + 7x^2 - 8x + 1 \)
22. \( f(x) = x^{10} - x^8 + x^6 - x^4 + x^2 - 1 \)

Find all of the zeros of each function.

23. \( g(x) = x^3 + 6x^2 + 21x + 26 \)
24. \( h(x) = x^3 - 6x^2 + 10x - 8 \)
25. \( f(x) = x^3 - 5x^2 - 7x + 51 \)
26. \( f(x) = x^3 - 7x^2 + 25x - 175 \)
27. \( g(x) = 2x^3 - x^2 + 28x + 51 \)
28. \( q(x) = 2x^3 - 17x^2 + 90x - 41 \)
29. \( h(x) = 4x^4 + 17x^2 + 4 \)
30. \( p(x) = x^4 - 9x^3 + 24x^2 - 6x - 40 \)
31. \( r(x) = x^4 - 6x^3 + 12x^2 + 6x - 13 \)
32. \( h(x) = x^4 - 15x^3 + 70x^2 - 70x - 156 \)

Write a polynomial function of least degree with integral coefficients that has the given zeros.

33. \(-4, 1, 5\)
34. \(-2, 2, 4, 6\)
35. \(4i, 3, -3\)
36. \(2i, 3i, 1\)
37. \(9, 1 + 2i\)
38. \(6, 2 + 2i\)

**PROFIT** For Exercises 39–41, use the following information.

A computer manufacturer determines that for each employee the profit for producing \(x\) computers per day is \(P(x) = -0.006x^4 + 0.15x^3 - 0.05x^2 - 1.8x\).

39. How many positive real zeros, negative real zeros, and imaginary zeros exist for this function? (Hint: Notice that 0, which is neither positive nor negative, is a zero of this function since \(d(0) = 0\).)

40. Approximate all real zeros to the nearest tenth by graphing the function using a graphing calculator.

41. What is the meaning of the roots in this problem?

**SPACE EXPLORATION** For Exercises 42 and 43, use the following information.

The space shuttle has an external tank for the fuel that the main engines need for the launch. This tank is shaped like a capsule, a cylinder with a hemispherical dome at either end. The cylindrical part of the tank has an approximate volume of \(336\pi\) cubic meters and a height of 17 meters more than the radius of the tank. (Hint: \(V(r) = \pi r^2h\))

42. Write an equation that represents the volume of the cylinder.

43. What are the dimensions of the cylindrical part of the tank?

**SCULPTING** For Exercises 44 and 45, use the following information.

Antonio is preparing to make an ice sculpture. He has a block of ice that he wants to reduce in size by shaving off the same amount from the length, width, and height. He wants to reduce the volume of the ice block to 24 cubic feet.

44. Write a polynomial equation to model this situation.

45. How much should he take from each dimension?
46. OPEN ENDED  Sketch the graph of a polynomial function that has the indicated number and type of zeros.
a. 3 real, 2 imaginary  b. 4 real  c. 2 imaginary

47. CHALLENGE  If a sixth-degree polynomial equation has exactly five distinct real roots, what can be said of one of its roots? Draw a graph of this situation.

48. REASONING  State the least degree a polynomial equation with real coefficients can have if it has roots at \( x = 5 + i \), \( x = 3 - 2i \), and a double root at \( x = 0 \). Explain.

49. CHALLENGE  Find a counterexample to disprove the following statement.
The polynomial function of least degree with integral coefficients with zeros at \( x = 4 \), \( x = -1 \), and \( x = 3 \), is unique.

50. Writing in Math  Use the information about medication on page 362 to explain how the roots of an equation can be used in pharmacology. Include an explanation of what the roots of this equation represent and an explanation of what the roots of this equation reveal about how often a patient should take this medication.

51. ACT/SAT  How many negative real zeros does \( f(x) = x^5 - 2x^4 - 4x^3 + 4x^2 - 5x + 6 \) have?
A 3  
B 2  
C 1  
D 0

52. REVIEW  Tiles numbered from 1 to 6 are placed in a bag and are drawn out to determine which of six tasks will be assigned to six people. What is the probability that the tiles numbered 5 and 6 are drawn consecutively?
F \( \frac{2}{3} \)  
G \( \frac{2}{5} \)  
H \( \frac{1}{2} \)  
J \( \frac{1}{3} \)

53. \( f(x) = x^3 - 5x^2 + 16x - 7 \)
54. \( f(x) = x^4 + 11x^3 - 3x^2 + 2x - 5 \)

Factor completely. If the polynomial is not factorable, write prime.
55. \( 15a^2b^2 - 5ab^2c^2 \)  
56. \( 12p^2 - 64p + 45 \)  
57. \( 4y^3 + 24y^2 + 36y \)

58. BASKETBALL  In a recent season, Monique Currie of the Duke Blue Devils scored 635 points. She made a total of 356 shots, including 3-point field goals, 2-point field goals, and 1-point free throws. She made 76 more 2-point field goals than free throws and 77 more free throws than 3-point field goals. Find the number of each type of shot she made.

PREREQUISITE SKILL  Find all values of \( \pm \frac{a}{b} \) given each replacement set.
59. \( a = \{1, 5\}; b = \{1, 2\} \)  
60. \( a = \{1, 2\}; b = \{1, 2, 7, 14\} \)  
61. \( a = \{1, 3\}; b = \{1, 3, 9\} \)  
62. \( a = \{1, 2, 4\}; b = \{1, 2, 4, 8, 16\} \)
On an airplane, carry-on baggage must fit into the overhead compartment above the passenger’s seat. The length of the compartment is 8 inches longer than the height, and the width is 5 inches shorter than the height. The volume of the compartment is 2772 cubic inches. You can solve the polynomial equation

\[(h + 8)(h - 5) = 2772,\]

where \(h\) is the height, \(h + 8\) is the length, and \(h - 5\) is the width, to find the dimensions of the overhead compartment.

**Identify Rational Zeros** Usually it is not practical to test all possible zeros of a polynomial function using only synthetic substitution. The Rational Zero Theorem can help you choose some possible zeros to test.

**Example**

List all of the possible rational zeros of each function.

\[f(x) = 2x^3 + 11x^2 + 12x + 9\]

If \(\frac{p}{q}\) is a rational zero, then \(p\) is a factor of 9 and \(q\) is a factor of 2. The possible values of \(p\) are \(±1, ±3, ±9\). The possible values for \(q\) are \(±1\) and \(±2\). So, \(\frac{p}{q} = ±1, ±3, ±9, ±\frac{1}{2}, ±\frac{3}{2}, \text{and} ±\frac{9}{2}\).

(continued on the next page)
b. \( f(x) = x^3 - 9x^2 - x + 105 \)

Since the coefficient of \( x^3 \) is 1, the possible rational zeros must be a factor of the constant term 105. So, the possible rational zeros are the integers \( \pm 1, \pm 3, \pm 5, \pm 7, \pm 15, \pm 21, \pm 35, \) and \( \pm 105. \)

**Find Rational Zeros** Once you have found the possible rational zeros of a function, you can test each number using synthetic substitution to determine the zeros of the function.

**EXAMPLE**

**Find Rational Zeros**

**GEOMETRY** The volume of a rectangular solid is 675 cubic centimeters. The width is 4 centimeters less than the height, and the length is 6 centimeters more than the height. Find the dimensions of the solid.

Let \( x \) = the height, \( x - 4 \) = the width, and \( x + 6 \) = the length.

Write an equation for the volume.

\[
\ell \cdot w \cdot h = V \\
(x - 4)(x + 6)x = 675 \\
x^3 + 2x^2 - 24x = 675 \\
x^3 + 2x^2 - 24x - 675 = 0
\]

The leading coefficient is 1, so the possible integer zeros are factors of 675, \( \pm 1, \pm 3, \pm 5, \pm 9, \pm 15, \pm 25, \pm 45, \pm 75, \pm 135, \pm 225, \) and \( \pm 675. \)

Since length can only be positive, we only need to check positive zeros. From Descartes’ Rule of Signs, we also know there is only one positive real zero. Make a table for the synthetic division and test possible real zeros.

One zero is 9. Since there is only one positive real zero, we do not have to test the other numbers. The other dimensions are 9 - 4 or 5 centimeters and 9 + 6 or 15 centimeters.

**CHECK** Verify that the dimensions are correct. \( 5 \times 9 \times 15 = 675 \checkmark \)

2. The volume of a rectangular solid is 1056 cubic inches. The length is 1 inch more than the width, and the height is 3 inches less than the width. Find the dimensions of the solid.
Find all of the zeros of \( f(x) = 2x^4 - 13x^3 + 23x^2 - 52x + 60. \)

From the corollary to the Fundamental Theorem of Algebra, we know there are exactly 4 complex roots. According to Descartes’ Rule of Signs, there are 4, 2, or 0 positive real roots and 0 negative real roots. The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 10, \pm 12, \pm 15, \pm 20, \pm 30, \pm 60, \pm \frac{1}{2}, \pm \frac{3}{2}, \pm \frac{5}{2}, \text{ and } \pm \frac{15}{2}. \)

Make a table and test some possible rational zeros.

<table>
<thead>
<tr>
<th>( p )</th>
<th>( q )</th>
<th>(-13)</th>
<th>( 23)</th>
<th>(-52)</th>
<th>(60)</th>
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<tr>
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<td>-11</td>
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<tr>
<td>5</td>
<td>2</td>
<td>-3</td>
<td>8</td>
<td>-12</td>
<td>0</td>
</tr>
</tbody>
</table>

Since \( f(5) = 0 \), you know that \( x = 5 \) is a zero. The depressed polynomial is \( 2x^3 - 3x^2 + 8x - 12 \).

Factor \( 2x^3 - 3x^2 + 8x - 12 \).

\[
\begin{align*}
2x^3 - 3x^2 + 8x - 12 &= 0 & \text{Write the depressed polynomial.} \\
2x^3 + 8x - 3x^2 - 12 &= 0 & \text{Regroup terms.} \\
2x(x^2 + 4) - 3(x^2 + 4) &= 0 & \text{Factor by grouping.} \\
(x^2 + 4)(2x - 3) &= 0 & \text{Distributive Property} \\
x^2 + 4 &= 0 & \text{or} & \ 2x - 3 &= 0 & \text{Zero Product Property} \\
x^2 &= -4 & \quad 2x &= 3 & \quad x &= \frac{3}{2} \\
x &= \pm 2i & \quad x &= \frac{3}{2} \\
\quad \text{There is another real zero at } x &= \frac{3}{2} \text{ and two imaginary zeros at } x = 2i \text{ and } x = -2i. \\
\quad \text{The zeros of this function are } 5, \frac{3}{2}, 2i \text{ and } -2i.
\]

Find all of the zeros of each function.

3A. \( h(x) = 9x^4 + 5x^2 - 4 \)

3B. \( k(x) = 2x^4 - 5x^3 + 20x^2 - 45x + 18 \)

List all of the possible rational zeros of each function.

1. \( p(x) = x^4 - 10 \)
2. \( d(x) = 6x^3 + 6x^2 - 15x - 2 \)

Find all of the rational zeros of each function.

3. \( p(x) = x^3 - 5x^2 - 22x + 56 \)
4. \( f(x) = x^3 - x^2 - 34x - 56 \)
5. \( l(x) = x^4 - 13x^2 + 36 \)
6. \( f(x) = 2x^3 - 7x^2 - 8x + 28 \)

7. **GEOMETRY** The volume of the rectangular solid is 1430 cubic centimeters. Find the dimensions of the solid.

Find all of the zeros of each function.

8. \( f(x) = 6x^3 + 5x^2 - 9x + 2 \)
9. \( f(x) = x^4 - x^3 - x^2 - x - 2 \)
List all of the possible rational zeros of each function.

10. \( f(x) = x^3 + 6x + 2 \)  
11. \( h(x) = x^3 + 8x + 6 \)  
12. \( f(x) = 3x^4 + 15 \)  
13. \( n(x) = x^5 + 6x^3 - 12x + 18 \)  
14. \( p(x) = 3x^3 - 5x^2 - 11x + 3 \)  
15. \( h(x) = 9x^6 - 5x^3 + 27 \)

Find all of the rational zeros of each function.

16. \( f(x) = x^3 + x^2 - 80x - 300 \)  
17. \( p(x) = x^3 - 3x - 2 \)  
18. \( f(x) = 2x^5 - x^4 - 2x + 1 \)  
19. \( f(x) = x^5 - 6x^3 + 8x \)  
20. \( g(x) = x^4 - 3x^3 + x^2 - 3x \)  
21. \( p(x) = x^4 + 10x^3 + 33x^2 + 38x + 8 \)

Find all of the zeros of each function.

22. \( p(x) = 6x^4 + 22x^3 + 11x^2 - 38x - 40 \)  
23. \( g(x) = 5x^4 - 29x^3 + 55x^2 - 28x \)  
24. \( h(x) = 6x^3 + 11x^2 - 3x - 2 \)  
25. \( p(x) = x^3 + 3x^2 - 25x + 21 \)  
26. \( h(x) = 10x^3 - 17x^2 - 7x + 2 \)  
27. \( g(x) = 48x^4 - 52x^3 + 13x - 3 \)  
28. \( p(x) = x^5 - 2x^4 - 12x^3 - 12x^2 - 13x - 10 \)  
29. \( h(x) = 9x^5 - 94x^3 + 27x^2 + 40x - 12 \)

**AUTOMOBILES** For Exercises 30 and 31, use the following information.

The length of the cargo space in a sport-utility vehicle is 4 inches greater than the height of the space. The width is sixteen inches less than twice the height. The cargo space has a total volume of 55,296 cubic inches.

30. Use a rectangular prism to model the cargo space. Write a polynomial function that represents the volume of the cargo space.

31. Will a package 34 inches long, 44 inches wide, and 34 inches tall fit in the cargo space? Explain.

**FOOD** For Exercises 32–34, use the following information.

A restaurant orders spaghetti sauce in cylindrical metal cans. The volume of each can is about 160\( \pi \) cubic inches, and the height of the can is 6 inches more than the radius.

32. Write a polynomial equation that represents the volume of a can. Use the formula for the volume of a cylinder, \( V = \pi r^2h \).

33. What are the possible values of \( r \)? Which values are reasonable here?

34. Find the dimensions of the can.

**AMUSEMENT PARKS** For Exercises 35–37, use the following information.

An amusement park owner wants to add a new wilderness water ride that includes a mountain that is shaped roughly like a square pyramid. Before building the new attraction, engineers must build and test a scale model.

35. If the height of the scale model is 9 inches less than its length, write a polynomial function that describes the volume of the model in terms of its length. Use the formula for the volume of a pyramid, \( V = \frac{1}{3}Bh \).

36. If the volume is 6300 cubic inches, write an equation for the situation.

37. What are the dimensions of the scale model?
For Exercises 38 and 39, use the following information.

38. Find all of the zeros of \( f(x) = x^3 - 2x^2 + 3 \) and \( g(x) = 2x^3 - 7x^2 + 2x + 3 \).
39. Determine which function, \( f \) or \( g \), is shown in the graph at the right.

**H.O.T. Problems:**

40. **FIND THE ERROR** Lauren and Luis are listing the possible rational zeros of \( f(x) = 4x^5 + 4x^4 - 3x^3 + 2x^2 - 5x + 6 \). Who is correct? Explain your reasoning.

41. **OPEN ENDED** Write a polynomial function that has possible rational zeros of \( \pm 1, \pm 3, \pm \frac{1}{2}, \pm \frac{3}{2} \).

42. **CHALLENGE** If \( k \) and \( 2k \) are zeros of \( f(x) = x^3 + 4x^2 + 9kx - 90 \), find \( k \) and all three zeros of \( f(x) \).

43. **Writing in Math** Use the information on page 369 to explain how the Rational Zero Theorem can be used to solve problems involving large numbers. Include the polynomial equation that represents the volume of the overhead baggage compartment and a list of all measures of the width of the compartment, assuming that the width is a whole number.

**STANDARDIZED TEST PRACTICE**

44. Which of the following is a zero of the function \( f(x) = 12x^3 - 5x^3 + 2x - 9 \)?
   A. \(-6\)
   B. \(\frac{3}{8}\)
   C. \(-\frac{2}{3}\)
   D. \(1\)

45. **REVIEW** A window is in the shape of an equilateral triangle. Each side of the triangle is 8 feet long. The window is divided in half by a support from one vertex to the midpoint of the side of the triangle opposite the vertex. Approximately how long is the support?
   F. 5.7 ft  
   H. 11.3 ft  
   G. 6.9 ft  
   J. 13.9 ft

**Spiral Review**

Given a function and one of its zeros, find all of the zeros of the function. (Lesson 6-8)

46. \( g(x) = x^3 + 4x^2 - 27x - 90; -3 \)
47. \( h(x) = x^3 - 11x + 20; 2 + i \)
48. \( f(x) = x^3 + 5x^2 + 9x + 45; -5 \)
49. \( g(x) = x^3 - 3x^2 - 41x + 203; -7 \)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials. (Lesson 6-7)

50. \( 20x^3 - 29x^2 - 25x + 6; x - 2 \)
51. \( 3x^4 - 21x^3 + 38x^2 - 14x + 24; x - 3 \)

52. **GEOMETRY** The perimeter of a right triangle is 24 centimeters. Three times the length of the longer leg minus two times the length of the shorter leg exceeds the hypotenuse by 2 centimeters. What are the lengths of all three sides? (Lesson 3-5)
CHAPTER 6
Study Guide
and Review

Key Concepts

Properties of Exponents (Lesson 6-1)
- The properties of powers for real numbers $a$ and $b$ and integers $m$ and $n$ are as follows.

\[
\frac{a^n}{b^n} \quad \text{if } b \neq 0
\]

\[
a^m \cdot a^n = a^{m+n}
\]

\[
(ab)^m = a^m b^m
\]

Operations with Polynomials (Lesson 6-2)
- To add or subtract: Combine like terms.
- To multiply: Use the Distributive Property.
- To divide: Use long division or synthetic division.

Polynomial Functions and Graphs (Lessons 6-4 and 6-5)
- Turning points of a function are called relative maxima and relative minima.

Solving Polynomial Equations (Lesson 6-6)
- You can factor polynomials using the GCF, grouping, or quadratic techniques.

The Remainder and Factor Theorems (Lesson 6-7)
- Factor Theorem: The binomial $x - a$ is a factor of the polynomial $f(x)$ if and only if $f(a) = 0$.

Roots, Zeros, and the Rational Zero Theorem (Lessons 6-8 and 6-9)
- Complex Conjugates Theorem: If $a + bi$ is a zero of a function, then $a - bi$ is also a zero.
- Integral Zero Theorem: If the coefficients of a polynomial function are integers such that $a_0 = 1$ and $a_n = 0$, any rational zeros of the function must be factors of $a_n$.

Key Vocabulary

degree of a polynomial (p. 320)
depressed polynomial (p. 357)
end behavior (p. 334)
leading coefficient (p. 331)
polynomial function (p. 332)
polynomial in one variable (p. 331)
quadratic form (p. 351)
relative maximum (p. 340)
relative minimum (p. 340)
scientific notation (p. 315)
simplify (p. 312)
standard notation (p. 315)
synthetic division (p. 327)
synthetic substitution (p. 356)

Vocabulary Check
Choose a term from the list above that best completes each statement or phrase.

1. A point on the graph of a polynomial function that has no other nearby points with lesser $y$-coordinates is a ____.

2. The _____ is the coefficient of the term in a polynomial function with the highest degree.

3. $(x^2)^2 - 17(x^2) + 16 = 0$ is written in ____.

4. A shortcut method known as ____ is used to divide polynomials by binomials.

5. A number is expressed in ________ when it is in the form $a \times 10^n$, where $1 \leq a < 10$ and $n$ is an integer.

6. The ________ is the sum of the exponents of the variables of a monomial.

7. When a polynomial is divided by one of its binomial factors, the quotient is called $a(n)$ _________.

8. When we ______ an expression, we rewrite it without parentheses or negative exponents.

9. What a graph does as $x$ approaches positive infinity or negative infinity is called the ________ of the graph.

10. The use of synthetic division to evaluate a function is called_______.

Vocabulary Review at algebra2.com
Lesson-by-Lesson Review

Properties of Exponents (pp. 312–318)

Simplify. Assume that no variable equals 0.

11. \( f^{-7} \cdot f^4 \)

12. \((3x^2)^3\)

13. \((2y)(4xy^3)\)

14. \(\left(\frac{3c^2}{5}\right)\left(\frac{4cd}{3}\right)^2\)

15. MARATHON Assume that there are 10,000 runners in a marathon and each runner runs a distance of 26.2 miles. If you add together the total number of miles for all runners, how many times around the world would the marathon runners have gone? Consider the circumference of Earth to be \(2.5 \times 10^4\) miles.

Example 1 Simplify \((3x^4y^6)(-8x^3y)\).

\[ (3x^4y^6)(-8x^3y) = (3)(-8)x^4 + 3y^6 + 1 \]

Commutative Property and Product of Powers

\[ = -24x^7y^7 \]

Simplify.

Example 2 Light travels at approximately \(3.0 \times 10^8\) meters per second. How far does light travel in one week?

Determine the number of seconds in one week.

\[ 60 \cdot 60 \cdot 24 \cdot 7 = 604,800 \text{ or } 6.048 \times 10^5 \text{ seconds} \]

Multiply by the speed of light.

\[ (3.0 \times 10^8) \cdot (6.048 \times 10^5) = 1.8144 \times 10^{14} \text{ m} \]

Operations with Polynomials (pp. 320–324)

Simplify.

16. \((4c - 5) - (c + 11) + (-6c + 17)\)

17. \((11x^2 + 13x - 15) - (7x^2 - 9x + 19)\)

18. \((d - 5)(d + 3)\)

19. \((2a^2 + 6)^2\)

20. CAR RENTAL The cost of renting a car is $40 per day plus $0.10 per mile. If a car is rented for \(d\) days and driven \(m\) miles a day, represent the cost \(C\).

Example 3 Find \((9k + 4)(7k - 6)\).

\[ (9k + 4)(7k - 6) = (9k)(7k) + (9k)(-6) + (4)(7k) + (4)(-6) \]

\[ = 63k^2 - 54k + 28k - 24 \]

\[ = 63k^2 - 26k - 24 \]

Dividing Polynomials (pp. 325–330)

Simplify.

21. \((2x^4 - 6x^3 + x^2 - 3x - 3) \div (x - 3)\)

22. \(x^4 + 18x^3 + 10x^2 + 3x) \div (x^2 + 3x)\)

23. SAILING The area of a triangular sail is \(16x^4 - 60x^3 - 28x^2 + 56x - 32\) square meters. The base of the triangle is \(x - 4\) meters. What is the height of the sail?

Example 4 Use synthetic division to find \((4x^4 - x^3 - 19x^2 + 11x - 2) \div (x - 2)\).

\[
\begin{array}{c|cccc}
2 & 4 & -1 & -19 & 11 & -2 \\
 & & 8 & -10 & 2 & \\
\hline
& 4 & 7 & -5 & 1 & \\
\end{array}
\]

The quotient is \(4x^3 + 7x^2 - 5x + 1\).
### 6–4 Polynomial Functions (pp. 331–338)

Find \( p(-4) \) and \( p(x + h) \) for each function.

24. \( p(x) = x - 2 \)
25. \( p(x) = -x + 4 \)
26. \( p(x) = 6x + 3 \)
27. \( p(x) = x^2 + 5 \)
28. \( p(x) = x^2 - x \)
29. \( p(x) = 2x^3 - 1 \)

30. **STORMS** The average depth of a tsunami can be modeled by 
\[
d(s) = \frac{s}{356} \sqrt{\frac{H^2}{20896}} - \frac{s^2}{20921} \],
\] where \( s \) is the speed in kilometers per hour and \( d \) is the average depth of the water in kilometers. Find the average depth of a tsunami when the speed is 250 kilometers per hour.

**Example 5** Find \( p(a + 1) \) if 
\[
p(x) = 5x - x^2 + 3x^3.
\]
\[
p(a + 1) = 5(a + 1) - (a + 1)^2 + 3(a + 1)^3
\]
\[
= 5a + 5 - (a^2 + 2a + 1) + 3(a^3 + 3a^2 + 3a + 1)
\]
\[
= 5a + 5 - a^2 - 2a - 1 + 3a^3 + 9a^2 + 9a + 3
\]
\[
= 3a^3 + 8a^2 + 12a + 7
\]

### 6–5 Analyzing Graphs of Polynomial Functions (pp. 339–347)

For Exercises 31–36, complete each of the following.

a. Graph each function by making a table of values.

b. Determine the consecutive integer values of \( x \) between which the real zeros are located.

c. Estimate the \( x \)-coordinates at which the relative maxima and relative minima occur.

31. \( h(x) = x^3 - 6x - 9 \)
32. \( f(x) = x^4 + 7x + 1 \)
33. \( p(x) = x^5 + x^4 - 2x^3 + 1 \)
34. \( g(x) = x^3 - x^2 + 1 \)
35. \( r(x) = 4x^3 + x^2 - 11x + 3 \)
36. \( f(x) = x^3 + 4x^2 + x - 2 \)

37. **PROFIT** A small business’ monthly profits for the first half of 2006 can be modeled by \((1, 550), (2, 725), (3, 680), (4, 830), (5, 920), (6, 810)\). How many turning points would the graph of a polynomial function through these points have? Describe them.
6–6

Solving Polynomial Equations  (pp. 349–355)

Factor completely. If the polynomial is not factorable, write prime.

38. \(10a^3 - 20a^2 - 2a + 4\)
39. \(5w^3 - 20w^2 + 3w - 12\)
40. \(x^4 - 7x^3 + 12x^2\)
41. \(x^2 - 7x + 5\)

Solve each equation.

42. \(3x^3 + 4x^2 - 15x = 0\)
43. \(m^4 + 3m^3 = 40m^2\)
44. \(x^4 - 8x^2 + 16 = 0\)
45. \(a^3 - 64 = 0\)

46. HOME DECORATING  The area of a dining room is 160 square feet. A rectangular rug placed in the center of the room is twice as long as it is wide. If the rug is bordered by 2 feet of hardwood floor on all sides, find the dimensions of the rug.

Example 7  Factor \(3m^2 + m - 4\).

Find two numbers with a product of \(3(-4)\) or \(-12\) and a sum of \(1\). The two numbers must be \(4\) and \(-3\) because \(4(-3) = -12\) and \(4 + (-3) = 1\).

\[
3m^2 + m - 4 = 3m^2 + 4m - 3m - 4 = (3m^2 + 4m) - (3m + 4) = m(3m + 4) + (-1)(3m + 4) = (3m + 4)(m - 1)
\]

Example 8  Solve \(x^3 - 3x^2 - 54x = 0\).

\[
x^3 - 3x^2 - 54x = 0
\]

\[
x(x - 9)(x + 6) = 0
\]

\[
x(x^2 - 3x - 54) = 0
\]

\[
x = 0 \text{ or } x - 9 = 0 \text{ or } x + 6 = 0
\]

\[
x = 0 \quad x = 9 \quad x = -6
\]

6–7

The Remainder and Factor Theorems  (pp. 356–361)

Use synthetic substitution to find \(f(3)\) and \(f(-2)\) for each function.

47. \(f(x) = x^2 - 5\)  48. \(f(x) = x^2 - 4x + 4\)
49. \(f(x) = x^3 - 3x^2 + 4x + 8\)

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.

50. \(x^3 + 5x^2 + 8x + 4; x + 1\)
51. \(x^3 + 4x^2 + 7x + 6; x + 2\)

52. PETS  The volume of water in a rectangular fish tank can be modeled by the polynomial \(3x^3 - x^2 - 34x - 40\). If the depth of the tank is given by the polynomial \(3x + 5\), what polynomials express the length and width of the fish tank?

Example 9  Show that \(x + 2\) is a factor of \(x^3 - 2x^2 - 5x + 6\). Then find any remaining factors of the polynomial.

\[
-2 \left|
\begin{array}{cccc}
1 & -2 & -5 & 6 \\
\hline
-2 & 8 & -6 \\
1 & -4 & 3 & 0
\end{array}
\right|
\]

The remainder is 0, so \(x + 2\) is a factor of \(x^3 - 2x^2 - 5x + 6\). Since \(x^3 - 2x^2 - 5x + 6 = (x + 2)(x^2 - 4x + 3)\), the remaining factors of \(x^3 - 2x^2 - 5x + 6\) are \(x - 3\) and \(x - 1\).
State the possible number of positive real zeros, negative real zeros, and imaginary zeros of each function.

53. \( f(x) = 2x^4 - x^3 + 5x^2 + 3x - 9 \)
54. \( f(x) = -4x^4 - x^2 - x + 1 \)
55. \( f(x) = 3x^4 - x^3 + 8x^2 + x - 7 \)
56. \( f(x) = 2x^4 - 3x^3 - 2x^2 + 3 \)

**DESIGN** For Exercises 57 and 58, use the following information.
An artist has a piece he wants displayed in a gallery. The gallery told him the biggest piece they would display is 72 cubic feet. The artwork is currently 5 feet long, 8 feet wide, and 6 feet high. Joe decides to cut off the same amount from the length, width, and height.

57. Assume that a rectangular prism is a good model for the artwork. Write a polynomial equation to model this situation.
58. How much should he take from each dimension?

Find all of the zeros of each function.

59. \( f(x) = 2x^3 - 13x^2 + 17x + 12 \)
60. \( f(x) = 3x^3 - 3x^2 - 10x + 24 \)
61. \( f(x) = x^4 - 4x^3 - 7x^2 + 34x - 24 \)
62. \( f(x) = 2x^3 - 5x^2 - 28x + 15 \)
63. \( f(x) = 2x^4 - 9x^3 + 2x^2 + 21x - 10 \)
64. **SHIPPING** The height of a shipping cylinder is 4 feet more than the radius. If the volume of the cylinder is \( 5\pi \) cubic feet, how tall is it? Use the formula \( V = \pi \cdot r^2 \cdot h \).

**Example 10** State the possible number of positive real zeros, negative real zeros, and imaginary zeros of \( f(x) = 5x^4 + 6x^3 - 8x + 12 \).

Since \( f(x) \) has two sign changes, there are 2 or 0 real positive zeros. 

\( f(-x) = 5x^4 - 6x^3 + 8x + 12 \)

Since \( f(-x) \) has two sign changes, there are 0 or 2 negative real zeros.

There are 0, 2, or 4 imaginary zeros.

**Example 11** Find all of the zeros of \( f(x) = x^3 + 7x^2 - 36 \).

There are exactly three complex zeros. There are one positive real zero and two negative real zeros. The possible rational zeros are \( \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36 \).

\[
\begin{array}{c|cccc}
2 & 1 & 7 & 0 & -36 \\
\hline 
 & 2 & 18 & 36 \\
1 & 9 & 18 & 0
\end{array}
\]

\( x^3 + 7x^2 - 36 = (x - 2)(x^2 + 9x + 18) = (x - 2)(x + 3)(x + 6) \)

Therefore, the zeros are 2, -3, and -6.
Simplify.
1. $(5b)^4(6c)^2$
2. $(13x - 1)(x + 3)$
3. $(3x^2 - 5x + 2) - (x^2 + 12x - 7)$
4. $(8x^3 + 9x^2 + 2x - 10) + (10x - 9)$
5. $(x^3 - x^2 - 10x^2 + 4x + 24) ÷ (x - 2)$
6. $(2x^3 + 9x^2 - 2x + 7) ÷ (x + 2)$

Given a polynomial and one of its factors, find the remaining factors of the polynomial. Some factors may not be binomials.
7. $x^3 - x^2 - 5x - 3; x + 1$
8. $x^3 + 8x + 24; x + 2$

Factor completely. If the polynomial is not factorable, write prime.
9. $3x^3y + x^2y^2 + x^2y$
10. $3x^2 - 2x - 2$
11. $ax^2 + 6ax + 9a$
12. $8r^3 - 64s^6$
13. $x^2 - 14x + 45$
14. $2r^2 + 3pr - 2p^2$

For Exercises 15–18, complete each of the following.
a. Graph each function by making a table of values.
b. Determine consecutive integer values of x between which each real zero is located.
c. Estimate the x-coordinates at which the relative maxima and relative minima occur.
15. $g(x) = x^3 + 6x^2 + 6x - 4$
16. $h(x) = x^4 + 6x^3 + 8x^2 - x$
17. $f(x) = x^3 + 3x^2 - 2x + 1$
18. $g(x) = x^4 - 2x^3 - 6x^2 + 8x + 5$

Solve each equation.
19. $a^4 = 6a^2 + 27$
20. $p^3 + 8p^2 = 18p$
21. $16x^4 - x^2 = 0$
22. $r^4 - 9r^2 + 18 = 0$
23. $p^2 - 8 = 0$
24. $n^3 + n - 27 = n$

25. **TRAVEL** While driving in a straight line from Milwaukee to Madison, your velocity is given by $v(t) = 5t^2 - 50t + 120$, where $t$ is driving time in hours. Estimate your speed after 1 hour of driving.

Use synthetic substitution to find $f(-2)$ and $f(3)$ for each function.
26. $f(x) = 7x^5 - 25x^4 + 17x^3 - 32x^2 + 10x - 22$
27. $f(x) = 3x^4 - 12x^3 - 21x^2 + 30x$
28. Write $36x^3 + 18x^3 + 5$ in quadratic form.
29. Write the polynomial equation of degree 4 with leading coefficient 1 that has roots at $-2, -1, 3$, and 4.

State the possible number of positive real zeros, negative real zeros, and imaginary zeros for each function.
30. $f(x) = x^3 - x^2 - 14x + 24$
31. $f(x) = 2x^3 - x^2 + 16x - 5$

Find all rational zeros of each function.
32. $g(x) = x^3 - 3x^2 - 53x - 9$
33. $h(x) = x^4 + 2x^3 - 23x^2 + 2x - 24$
34. $f(x) = 5x^3 - 29x^2 + 55x - 28$
35. $g(x) = 4x^3 + 16x^2 - x - 24$

**FINANCIAL PLANNING** For Exercises 36 and 37, use the following information.
Toshi will start college in six years. According to their plan, Toshi’s parents will save $1000 each year for the next three years. During the fourth and fifth years, they will save $1200 each year. During the last year before he starts college, they will save $2000.

36. In the formula $A = P(1 + r)^t$, $A =$ the balance, $P =$ the amount invested, $r =$ the interest rate, and $t =$ the number of years the money has been invested. Use this formula to write a polynomial equation to describe the balance of the account when Toshi starts college.
37. Find the balance of the account if their investment yields 6% annually.
Read each question. Then fill in the correct answer on the answer document provided by your teacher or on a sheet of paper.

1. Which expression is equivalent to $3a(2a + 1) - (2a - 2)(a + 3)$?
   A $2a^2 + 6a + 7$
   B $4a^2 - a + 6$
   C $4a^2 + 6a - 6$
   D $4a^2 - 3a + 7$

2. The figure below shows the first 3 stages of a fractal.

   How many rectangles will the $n$th stage of this fractal contain?
   F $2n$
   G $2^n$
   H $2n - 1$
   J $2^n - 1$

3. GRIDDABLE Miguel is finding the perimeter of the quadrilateral below. What is the value of the constant term of the perimeter?

   A $3x^2 - 14x + 8$
   B $3x^2 + 14x + 8$
   C $3x^2 - 8$
   D $4x + 6$
7. The figure below is the net of a rectangular prism. Use a ruler to measure the dimensions of the net to the nearest tenth of a centimeter.

Which measurement best approximates the volume of the rectangular prism represented by the net?

F 6.3 cm³
G 10.5 cm³
H 26.3 cm³
J 44.3 cm³

8. Which of the following is a true statement about the cube whose net is shown below?

A Faces L and M are parallel.
B Faces N and O are parallel.
C Faces M and P are perpendicular.
D Faces Q and L are perpendicular.

9. Kelly is designing a 12-inch by 12-inch scrapbook page. She cuts one picture that is 4 inches by 6 inches. She decides that she wants the next picture to be 75% as big as the first picture and the third picture to be 150% larger than the second picture. What are the approximate dimensions of the third picture?

F 0.45 in. by 0.68 in.
G 3.0 in. by 4.5 in.
H 4.5 in. by 6.75 in.
J 6.0 in. by 9.0 in.

10. GRIDDABLE Jalisa is a waitress. She recorded the following data about the amount that she made in tips for a certain number of hours.

<table>
<thead>
<tr>
<th>Amount of Tips</th>
<th>Hours Worked</th>
</tr>
</thead>
<tbody>
<tr>
<td>$12</td>
<td>1</td>
</tr>
<tr>
<td>$36</td>
<td>3</td>
</tr>
<tr>
<td>$60</td>
<td>5</td>
</tr>
</tbody>
</table>

If Jalisa continues to make the same amount of tips as shown in the table above, how much, in dollars, will she make in tips for working 9 hours?

Record your answers on a sheet of paper. Show your work.

11. Consider the polynomial function \( f(x) = 3x^4 + 19x^3 + 7x^2 - 11x - 2 \).

a. What is the degree of the function?

b. What is the leading coefficient of the function?

c. Evaluate \( f(1), f(-2), \) and \( f(2n) \). Show your work.